Tetrad Gravity: I) A New Formulation.

Luca Lusanna
Sezione INFN di Firenze
L.go E.Fermi 2 (Arcetri)
50125 Firenze, Italy
E-mail LUSANNA@FI.INFN.IT

and

Stefano Russo
Condominio dei Pioppi 16
6916 Grancia (Lugano)
Svizzera

Abstract

A new version of tetrad gravity in globally hyperbolic, asymptotically flat at spatial infinity spacetimes with Cauchy surfaces diffeomorphic to R^3 is obtained by using a new parametrization of arbitrary cotetrads to define a set of configurational variables to be used in the ADM metric action. Seven of the fourteen first class constraints have the form of the vanishing of canonical momenta. A comparison is made with other models of tetrad gravity and with the ADM canonical formalism for metric gravity. The phase space expression of various 4-tensors is explicitly given.

February 7, 2008

This work has been partially supported by the network "Constrained Dynamical Systems" of the E.U. Programme "Human Capital and Mobility".

Typeset using REVTEX

I. INTRODUCTION

This is the first of a series of papers on the canonical reduction of a new formulation of tetrad gravity, motivated by the attempt to arrive at a unified description of the four interactions [with the matter being either Grassmann-valued Dirac fields or relativistic particles] based on Dirac-Bergmann theory of constraints, which is needed for the Hamiltonian formulation of both gauge theories and general relativity. Therefore, we shall study general relativity from the canonical point of view generalizing to it all the results already obtained in the canonical study of gauge theories in a systematic way, since neither a complete reduction of gravity with an identification of the physical canonical degrees of freedom of the gravitational field nor a detailed study of its Hamiltonian group of gauge transformations (whose infinitesimal generators are the first class constraints) has ever been pushed till the end in an explicit way.

The research program aiming to express the special relativistic strong, weak and electromagnetic interactions in terms of Dirac's observables [1] is in an advanced stage of development [2]. This program is based on the Shanmugadhasan canonical transformations [3]: if a system has 1st class constraints at the Hamiltonian level (so that its dynamics is restricted to a presymplectic submanifold of phase space), then, at least locally, one can find a canonical basis with as many new momenta as 1st class constraints (Abelianization of 1st class constraints), with their conjugate canonical variables as Abelianized gauge variables and with the remaining pairs of canonical variables as pairs of canonically conjugate Dirac's observables (canonical basis of physical variables adapted to the chosen Abelianization; they give a trivialization of the BRST construction of observables). Putting equal to zero the Abelianized gauge variables one defines a local gauge of the model. If a system with constraints admits one (or more) global Shanmugadhasan canonical transformations, one obtains one (or more) privileged global gauges in which the physical Dirac observables are globally defined and globally separated from the gauge degrees of freedom [for systems with a compact configuration space this is impossible]. These privileged gauges (when they exist) can be called generalized Coulomb gauges. Second class constraints, when present, are also taken into account by the Shanmugadhasan canonical transformation [3].

Firstly, inspired by Ref. [4], the canonical reduction to noncovariant generalized Coulomb gauges, with the determination of the physical Hamiltonian as a function of a canonical basis of Dirac's observables, has been achieved for the following isolated systems (for them one can ask that the 10 conserved generators of the Poincaré algebra are finite so to be able to use group theory; theories with external fields can only be recovered as limits in some parameter of a subsystem of the isolated system):

a) Yang-Mills theory with Grassmann-valued fermion fields [5] in the case of a trivial principal bundle over a fixed-x^o R^3 slice of Minkowski spacetime with suitable Hamiltonian-oriented boundary conditions; this excludes monopole solutions and, since R^3 is not compactified, one has only winding number and no instanton number. After a discussion of the Hamiltonian formulation of Yang-Mills theory, of its group of gauge transformations and of the Gribov ambiguity, the theory has been studied in suitable weighted Sobolev spaces where the Gribov ambiguity is absent and the global color charges are well defined. The global Dirac observables are the transverse quantities $\vec{A}_{a\perp}(\vec{x}, x^o)$, $\vec{E}_{a\perp}(\vec{x}, x^o)$ and fermion fields dressed with Yang-Mills (gluonic) clouds. The nonlocal and nonpolynomial (due to the presence of classical Wilson lines along flat geodesics) physical Hamiltonian has been obtained: it is nonlocal but without any kind of singularities, it has the correct Abelian limit if the structure constants are turned off, and it contains the explicit realization of the abstract Mitter-Viallet metric.

b) The Abelian and non-Abelian SU(2) Higgs models with fermion fields [6,7], where the symplectic decoupling is a refinement of the concept of unitary gauge. There is an ambiguity in the solutions of the Gauss law constraints, which reflects the existence of disjoint sectors of solutions of the Euler-Lagrange equations of Higgs models. The physical Hamiltonian and Lagrangian of the Higgs phase have been found; the self-energy turns out to be local and contains a local four-fermion interaction.

c) The standard SU(3)xSU(2)xU(1) model of elementary particles [8] with Grassmann-valued fermion fields. The final reduced Hamiltonian contains nonlocal self-energies for the electromagnetic and color interactions, but "local ones" for the weak interactions implying the nonperturbative emergence of 4-fermions interactions.

The next problem is how to covariantize these results. Again the starting point was given by Dirac [1] with his reformulation of classical field theory on spacelike hypersurfaces foliating Minkowski spacetime M^4 [the foliation is defined by an embedding $R \times \Sigma \rightarrow M^4$, $(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma})$, with Σ an abstract 3-surface diffeomorphic to R^3: this is the classical basis of Tomonaga-Schwinger quantum field theory]. In this way one gets parametrized field theory with a covariant 3+1 splitting of flat spacetime and already in a form suited to the transition to general relativity in its ADM canonical formulation (see also Ref. [9] , where a theoretical study of this problem is done in curved spacetimes). The price is that one has to add as new configuration variables the points $z^\mu(\tau, \vec{\sigma})$ of the spacelike hypersurface Σ_τ [the only ones carrying Lorentz indices; the scalar parameter τ labels the leaves of the foliation and $\vec{\sigma}$ are curvilinear coordinates on Σ_τ] and then to define the fields on Σ_τ so that they know the hypersurface Σ_τ of τ-simultaneity [for a Klein-Gordon field $\phi(x)$, this new field is $\tilde{\phi}(\tau, \vec{\sigma}) = \phi(z(\tau, \vec{\sigma}))$: it contains the nonlocal information about the embedding]. Then one rewrites the Lagrangian of the given isolated system in the form required by the coupling to an external gravitational field, makes the previous 3+1 splitting of Minkowski spacetime and interpretes all the fields of the system as the new fields on Σ_τ (they are Lorentz scalars, having only surface indices). Instead of considering the 4-metric as describing a gravitational field (and therefore as an independent field as it is done in metric gravity, where one adds the Hilbert action to the action for the matter fields), here one replaces the 4-metric with the the induced metric $g_{AB}[z] = z^{(\mu)}_A \eta_{(\mu)(\nu)} z^{(\nu)}_B$ on Σ_τ [a functional of $z^{(\mu)}$; here we use the notation $\sigma^A = (\tau, \sigma^r)$; $z^{(\mu)}_A = \partial z^{(\mu)}/\partial \sigma^A$ are flat tetrad fields on Minkowski spacetime with the $z^{(\mu)}_r$'s tangent to Σ_τ] and considers the embedding coordinates $z^{(\mu)}(\tau, \vec{\sigma})$ as independent fields [this is not possible in metric gravity, because in curved spacetimes $z^\mu_A \neq \partial z^\mu/\partial \sigma^A$ are not tetrad fields since the holonomic coordinates $z^\mu(\tau, \vec{\sigma})$ do not exist]. From this Lagrangian, besides a Lorentz-scalar form of the constraints of the given system, we get four extra primary first class constraints $\mathcal{H}_\mu(\tau, \vec{\sigma}) \approx 0$ implying the independence of the description from the choice of the foliation with spacelike hypersufaces. In special relativity, it is convenient to restrict ourselves to arbitrary spacelike hyperplanes $z^\mu(\tau, \vec{\sigma}) = x^\mu_s(\tau) + b^\mu_{\check{r}}(\tau)\sigma^{\check{r}}$. Since they are described by only 10 variables [an origin $x^\mu_s(\tau)$ and, on it, three orthogonal spacelike unit vectors generating the fixed constant timelike unit normal to the hyperplane], we remain only with 10 first class constraints determining the 10 variables conjugate to the hyperplane [they are a 4-momentum p^μ_s and the six independent degrees of freedom hidden in a spin tensor $S^{\mu\nu}_s$] in terms of the variables of the system.

If we now consider only the set of configurations of the isolated system with timelike ($p^2_s > 0$) 4-momenta, we can restrict the description to the so-called Wigner hyperplanes orthogonal to p^μ_s itself. To get this result, we must boost at rest all the variables with Lorentz indices by using the standard Wigner boost $L^\mu{}_\nu(p_s, \overset{\circ}{p}_s)$ for timelike Poincaré orbits, and then add the gauge-fixings $b^\mu_{\check r}(\tau) - L^\mu{}_{\check r}(p_s, \overset{\circ}{p}_s) \approx 0$. Since these gauge-fixings depend on p^μ_s, the final canonical variables, apart p^μ_s itself, are of 3 types: i) there is a non-covariant "external" center-of-mass variable $\tilde x^\mu(\tau)$ [it is only covariant under the little group of timelike Poincaré orbits like the Newton-Wigner position operator]; ii) all the 3-vector variables become Wigner spin 1 3-vectors [boosts in M^4 induce Wigner rotations on them]; iii) all the other variables are Lorentz scalars. Only four 1st class constraints are left. One obtains in this way a new kind of instant form of the dynamics (see Ref. [10]), the "Wigner-covariant 1-time rest-frame instant form" [11] with a universal breaking of Lorentz covariance. It is the special relativistic generalization of the nonrelativistic separation of the center of mass from the relative motions [$H = \frac{\vec P^2}{2M} + H_{rel}$]. The role of the "external" center of mass is taken by the point $\tilde x^\mu(\tau)$ in the Wigner hyperplane and by its normal p^μ_s. The four 1st class constraints can be put in the following form: i) the vanishing of the total (Wigner spin 1) 3-momentum of the system $\vec p_{sys} \approx 0$, saying that the Wigner hyperplane $\Sigma_{W\,\tau}$ is the intrinsic rest frame [instead, $\vec p_s$ is left arbitrary, since p^μ_s depends upon the orientation of the Wigner hyperplane with respect to arbitrary reference frames in M^4]; ii) $\pm\sqrt{p^2_s} - M_{sys} \approx 0$, saying that the invariant mass M_{sys} of the system replaces the nonrelativistic Hamiltonian H_{rel} for the relative degrees of freedom, after the addition of the gauge-fixing $T_s - \tau \approx 0$ [identifying the time parameter τ, labelling the leaves of the foliation, with the Lorentz scalar time of the center of mass in the rest frame, $T_s = p_s \cdot \tilde x_s / M_{sys}$; M_{sys} generates the evolution in this time].

Now 3 degrees of freedom of the isolated system [an "internal" center-of-mass 3-variable $\vec\sigma_{sys}$ defined inside the Wigner hyperplane and conjugate to $\vec p_{sys}$] become gauge variables [the natural gauge fixing is $\vec\sigma_{sys} \approx 0$, so that it coincides with the origin $x^{(\mu)}_s(\tau) = z^{(\mu)}(\tau, \vec\sigma = 0)$ of the Wigner hyperplane], while the $\tilde x^{(\mu)}$ is playing the role of a kinematical external center of mass for the isolated system and may be interpreted as a decoupled observer with his parametrized clock (point particle clock). All the fields living on the Wigner hyperplane are now either Lorentz scalar or with their 3-indices transformaing under Wigner rotations (induced by Lorentz transformations in Minkowski spacetime) as any Wigner spin 1 index. The determination of $\vec\sigma_{sys}$ may be done with the group theoretical methods of Ref. [12]: given a realization on the phase space of a given system of the ten Poincaré generators one can build three 3-position variables only in terms of them, which in our case of a system on the Wigner hyperplane with $\vec p_{sys} \approx 0$ are: i) a canonical center of mass (the "internal" center of mass $\vec\sigma_{sys}$); ii) a noncanonical Møller center of energy $\vec\sigma^{(E)}_{sys}$; iii) a noncanonical Fokker-Pryce center of inertia $\vec\sigma^{(FP)}_{sys}$. Due to $\vec p_{sys} \approx 0$, we have $\vec\sigma_{sys} \approx \vec\sigma^{(E)}_{sys} \approx \vec\sigma^{(FP)}_{sys}$. By adding the gauge fixings $\vec\sigma_{sys} \approx 0$ one can show that the origin $x^{(\mu)}_s(\tau)$ becomes simultaneously the Dixon center of mass of an extended object and both the Pirani and Tulczyjew centroids (see Ref. [15,16] for the application of these methods to find the center of mass of a configuration of the Klein-Gordon field after the preliminary work of Ref. [14]). With similar methods one can construct three "external" collective positions (all located on the Wigner hyperplane): i) the "external" canonical noncovariant center of mass $\tilde{x}_s^{(\mu)}$; ii) the "external" noncanonical and noncovariant Møller center of energy $R_s^{(\mu)}$; iii) the "external" covariant noncanonical Fokker-Pryce center of inertia $Y_s^{(\mu)}$ (when there are the gauge fixings $\vec{\sigma}_{sys} \approx 0$ it also coincides with the origin $x_s^{(\mu)}$). It turns out that the Wigner hyperplane is the natural setting for the study of the Dixon multipoles of extended relativistic systems [13] and for defining the canonical relative variables with respect to the center of mass. After having put control on the relativistic definitions of center of mass of an extended system, the lacking kinematics of relativistic rotations in now under investigation. The Wigner hyperplane with its natural Euclidean metric structure offers a natural solution to the problem of boost for lattice gauge theories and realizes explicitly the machian aspect of dynamics that only relative motions are relevant.

The isolated systems till now analyzed to get their rest-frame Wigner-covariant generalized Coulomb gauges [i.e. the subset of global Shanmugadhasan canonical bases, which, for each Poincaré stratum, are also adapted to the geometry of the corresponding Poincaré orbits with their little groups; these special bases can be named Poincaré-Shanmugadhasan bases for the given Poincaré stratum of the presymplectic constraint manifold (every stratum requires an independent canonical reduction); till now only the main stratum with P^2 timelike and $W^2 \neq 0$ has been investigated] are:

a) The system of N scalar particles with Grassmann electric charges plus the electromagnetic field [11]. The starting configuration variables are a 3-vector $\vec{\eta}_i(\tau)$ for each particle $[x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))]$ and the electromagnetic gauge potentials $A_{\check{A}}(\tau, \vec{\sigma}) = \frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial \sigma^{\check{A}}} A_\mu(z(\tau, \vec{\sigma}))$, which know the embedding of Σ_τ into M^4. One has to choose the sign of the energy of each particle, because there are not mass-shell constraints (like $p_i^2 - m_i^2 \approx 0$) among the constraints of this formulation, due to the fact that one has only three degrees of freedom for particle, determining the intersection of a timelike trajectory and of the spacelike hypersurface Σ_τ. For each choice of the sign of the energy of the N particles, one describes only one of the branches of the mass spectrum of the manifestly covariant approach based on the coordinates $x_i^\mu(\tau)$, $p_i^\mu(\tau)$, i=1,..,N, and on the constraints $p_i^2 - m_i^2 \approx 0$ (in the free case). In this way, one gets a description of relativistic particles with a given sign of the energy with consistent couplings to fields and valid independently from the quantum effect of pair production [in the manifestly covariant approach, containing all possible branches of the particle mass spectrum, the classical counterpart of pair production is the intersection of different branches deformed by the presence of fields]. The final Dirac's observables are: i) the transverse radiation field variables; ii) the particle canonical variables $\vec{\eta}_i(\tau)$, $\check{\vec{\kappa}}_i(\tau)$, dressed with a Coulomb cloud. The physical Hamiltonian contains the mutual instantaneous Coulomb potentials extracted from field theory and there is a regularization of the Coulomb self-energies due to the Grassmann character of the electric charges Q_i $[Q_i^2 = 0]$. In Ref. [17] there is the study of the Lienard-Wiechert potentials and of Abraham-Lorentz-Dirac equations in this rest-frame Coulomb gauge and also scalar electrodynamics is reformulated in it. Also the rest-frame 1-time relativistic statistical mechanics has been developed [11].

b) The system of N scalar particles with Grassmann-valued color charges plus the color SU(3) Yang-Mills field [18]: it gives the pseudoclassical description of the relativistic scalar-quark model, deduced from the classical QCD Lagrangian and with the color field present. The physical invariant mass of the system is given in terms of the Dirac observables. From the reduced Hamilton equations the second order equations of motion both for the reduced transverse color field and the particles are extracted. Then, one studies the N=2 (meson) case. A special form of the requirement of having only color singlets, suited for a field-independent quark model, produces a "pseudoclassical asymptotic freedom" and a regularization of the quark self-energy. With these results one can covariantize the bosonic part of the standard model given in Ref. [8].

c) The system of N spinning particles of definite energy $[(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ representation of SL(2,C)] with Grassmann electric charges plus the electromagnetic field [19] and that of a Grassmann-valued Dirac field plus the electromagnetic field (the pseudoclassical basis of QED) [20]. In both cases there are geometrical complications connected with the spacetime description of the path of electric currents and not only of their spin structure, suggesting a reinterpretation of the supersymmetric scalar multiplet as a spin fibration; a new canonical decomposition of the Klein-Gordon field into collective and relative variables [14,16] will be helpful to clarify these problems. After their solution and after having obtained the description of Grassmann-valued chiral fields [this will require the transcription of the front form of the dynamics in the instant one for the Poincaré strata with $P^2=0$] the rest-frame form of the full standard $SU(3)\times SU(2)\times U(1)$ model can be achieved.

Finally, to eliminate the three 1st class constraints $\vec{p}[system]\approx 0$ by finding their natural gauge-fixings, when fields are present, one needs to find a rest-frame canonical basis of center-of-mass and relative variables for fields (in analogy to particles). A basis with a "center of phase" has already been found for a real Klein-Gordon field both in the covariant approach [14] and on spacelike hypersurfaces [16]. In this case also the "internal" center of mass has been found, but not yet a canonical basis containing it. There is the hope that all these new pieces of information will allow, after quantization of this new consistent relativistic mechanics without the classical problems connected with pair production, to find the asymptotic states of the covariant Tomonaga-Schwinger formulation of quantum field theory on spacelike hypersurfaces: these states are needed for the theory of quantum bound states [since Fock states do not constitute a Cauchy problem for the field equations, because an in (or out) particle can be in the absolute future of another one due to the tensor product nature of these asymptotic states, bound state equations like the Bethe-Salpeter one have spurious solutions which are excitations in relative energies, the variables conjugate to relative times (which are gauge variables [11])]. Moreover, it will be possible to include bound states among the asymptotic states.

As said in Ref. [17,18], the quantization of these rest-frame models has to overcome two problems. On the particle side, the complication is the quantization of the square roots associated with the relativistic kinetic energy terms: in the free case this has been done in Ref. [21] [see Refs. [22] for the complications induced by the Coulomb potential]. On the field side (all physical Hamiltonian are nonlocal and, with the exception of the Abelian case, nonpolynomial, but quadratic in the momenta), the obstacle is the absence (notwithstanding there is no no-go theorem) of a complete regularization and renormalization procedure of electrodynamics (to start with) in the Coulomb gauge: see Ref. [23] (and its bibliography) for the existing results for QED.

However, as shown in Refs. [11,5] [see their bibliography for the relevant references regarding all the quantities introduced in this Section], the rest-frame instant form of dynamics automatically gives a physical ultraviolet cutoff in the spirit of Dirac and Yukawa: it is the Møller radius [24] $\rho=\sqrt{-W^2}/P^2=|\vec{S}|/\sqrt{P^2}$ ($W^2=-P^2\vec{S}^2$ is the Pauli-Lubanski

Casimir when $P^2 > 0$), namely the classical intrinsic radius of the worldtube, around the covariant noncanonical Fokker-Pryce center of inertia Y^μ, inside which the noncovariance of the canonical center of mass $\tilde{x}^\mu$ is concentrated. At the quantum level ρ becomes the Compton wavelength of the isolated system multiplied its spin eigenvalue $\sqrt{s(s+1)}$, $\rho \mapsto \hat{\rho} = \sqrt{s(s+1)}\hbar/M = \sqrt{s(s+1)}\lambda_M$ with $M = \sqrt{P^2}$ the invariant mass and $\lambda_M = \hbar/M$ its Compton wavelength. Therefore, the criticism to classical relativistic physics, based on quantum pair production, concerns the testing of distances where, due to the Lorentz signature of spacetime, one has intrinsic classical covariance problems: it is impossible to localize the canonical center of mass $\tilde{x}^\mu$ adapted to the first class constraints of the system (also named Pryce center of mass and having the same covariance of the Newton-Wigner position operator) in a frame independent way.

Let us remember [11] that ρ is also a remnant in flat Minkowski spacetime of the energy conditions of general relativity: since the Møller noncanonical, noncovariant center of energy has its noncovariance localized inside the same worldtube with radius ρ (it was discovered in this way) [24], it turns out that for an extended relativistic system with the material radius smaller than its intrinsic radius ρ one has: i) its peripheral rotation velocity can exceed the velocity of light; ii) its classical energy density cannot be positive definite everywhere in every frame.

Now, the real relevant point is that this ultraviolet cutoff determined by ρ exists also in Einstein's general relativity (which is not power counting renormalizable) in the case of asymptotically flat spacetimes, taking into account the Poincaré Casimirs of its asymptotic ADM Poincaré charges (when supertranslations are eliminated with suitable boundary conditions; let us remark that Einstein and Wheeler use closed universes because they don't want to introduce boundary conditions, but in this way they loose Poincaré charges and the possibility to make contact with particle physics and to define spin). The generalization of the worldtube of radius ρ to asymptotically flat general relativity with matter could be connected with the unproved cosmic censorship hypothesis.

Moreover, the extended Heisenberg relations of string theory [25], i.e. $\triangle x = \frac{\hbar}{\triangle p} + \frac{\triangle p}{T_{cs}} = \frac{\hbar}{\triangle p} + \frac{\hbar \triangle p}{L_{cs}^2}$ implying the lower bound $\triangle x > L_{cs} = \sqrt{\hbar/T_{cs}}$ due to the $y + 1/y$ structure, have a counterpart in the quantization of the Møller radius [11]: if we ask that, also at the quantum level, one cannot test the inside of the worldtube, we must ask $\triangle x > \hat{\rho}$ which is the lower bound implied by the modified uncertainty relation $\triangle x = \frac{\hbar}{\triangle p} + \frac{\hbar \triangle p}{\hat{\rho}^2}$. This could imply that the center-of-mass canonical noncovariant 3-coordinate $\vec{z} = \sqrt{P^2}(\vec{\tilde{x}} - \frac{\vec{P}}{P^o}\tilde{x}^o)$ [11] cannot become a self-adjoint operator. See Hegerfeldt's theorems (quoted in Refs. [5,11]) and his interpretation pointing at the impossibility of a good localization of relativistic particles (experimentally one determines only a worldtube in spacetime emerging from the interaction region). Since the eigenfunctions of the canonical center-of-mass operator are playing the role of the wave function of the universe, one could also say that the center-of-mass variable has not to be quantized, because it lies on the classical macroscopic side of Copenhagen's interpretation and, moreover, because, in the spirit of Mach's principle that only relative motions can be observed, no one can observe it (it is only used to define a decoupled "point particle clock"). On the other hand, if one rejects the canonical noncovariant center of mass in favor of the covariant noncanonical Fokker-Pryce center of inertia Y^μ, $\{Y^\mu, Y^\nu\} \neq 0$, one could invoke the philosophy of quantum groups to quantize Y^μ to get some kind of quantum plane for the center-of-mass description. Let us remark that the quantization of the square root Hamiltonian done in Ref. [21] is consistent with this problematic.

In conclusion, the best set of canonical coordinates adapted to the constraints and to the geometry of Poincaré orbits and naturally predisposed to the coupling to canonical tetrad gravity is emerging for the electromagnetic, weak and strong interactions with matter described either by fermion fields or by relativistic particles with a definite sign of the energy. Therefore, one can begin to think how to quantize the standard model in the Wigner-covariant Coulomb gauge in the rest-frame instant form (the classical background fo the Tomonaga-Schwinger approach to quantum field theory) with the Möller radius as a ultraviolet cutoff.

Since our aim is to arrive at a unified description of the four interactions, in this paper and in the following ones we shall explore the canonical reduction to Dirac's observables of tetrad gravity (more natural than metric gravity for the coupling to fermion fields) and we shall begin to explore the connection of Dirac's observables with Bergmann's definition of observables and the problem of time in general relativity [26–28]. Moreover, in globally hyperbolic, asymptotically flat at spatial infinity, spacetimes, we shall arrive at a solution of the deparametrization problem of general relativity (how to recover the rest-frame instant form when the Newton constant is put equal to zero, G=0), to a solution, till now at order G, of the superhamiltonian constraint, with the matter represented (to start with) by N massive scalar particles, allowing to visualize the instantaneous part of the interaction (think to the Coulomb potential in the electromagnetic Coulomb gauge), and to the identification of the volume expression of the ADM energy as the reduced Hamiltonian of the universe, containing all the interactions. Then, the replacement of scalar particles with spinning ones will allow to test the precessional effects (gravitomagnetism) of general relativity.

We shall restrict ourselves to the simplest class of spacetimes to have some chance to arrive at the end of the canonical reduction. Refs. [29–31] are used for the background in differential geometry. A spacetime is a time-oriented pseudo-Riemannian (or Lorentzian) 4-manifold $(M^4, {}^4g)$ with signature $\epsilon\, (+---)$ $(\epsilon = \pm 1)$ and with a choice of time orientation [i.e. there exists a continuous, nowhere vanishing timelike vector field which is used to separate the nonspacelike vectors at each point of M^4 in either future- or past-directed vectors]. Our spacetimes are assumed to be:

i) Globally hyperbolic 4-manifolds, i.e. topologically they are $M^4 = R \times \Sigma$, so to have a well posed Cauchy problem [with Σ the abstract model of Cauchy surface] at least till when no singularity develops in M^4 [see the singularity theorems]. Therefore, these spacetimes admit regular foliations with orientable, complete, non-intersecting spacelike 3-manifolds: the leaves of the foliation are the embeddings $i_\tau : \Sigma \rightarrow \Sigma_\tau \subset M^4$, $\vec{\sigma} \mapsto z^\mu(\tau, \vec{\sigma})$, where $\vec{\sigma} = \{\sigma^r\}$, r=1,2,3, are local coordinates in a chart of the C^∞-atlas of the abstract 3-manifold Σ and $\tau : M^4 \rightarrow R$, $z^\mu \mapsto \tau(z^\mu)$, is a global timelike future-oriented function labelling the leaves (surfaces of simultaneity). In this way, one obtains 3+1 splittings of M^4 and the possibility of a Hamiltonian formulation.

ii) Asymptotically flat at spatial infinity, so to have the possibility to define asymptotic Poincaré charges [32–37]: they allow the definition of a Møller radius in general relativity and are a bridge towards a future soldering with the theory of elementary particles in Minkowski spacetime defined as irreducible representation of its kinematical, globally implemented Poincaré group according to Wigner. In this paper we will not compactify space infinity at a point like in the spi approach of Ref. [37].

iii) Since we want to be able to introduce Dirac fermion fields, our spacetimes M^4 must admit a spinor (or spin) structure [38]. Since we consider noncompact space- and time-orientable spacetimes, spinors can be defined if and only if they are "parallelizable" [39]. This means that we have trivial principal frame bundle $L(M^4) = M^4 \times GL(4,R)$ with GL(4,R) as structure group and trivial orthonormal frame bundle $F(M^4) = M^4 \times SO(3,1)$; the fibers of $F(M^4)$ are the disjoint union of four components and $F_o(M^4) = M^4 \times L_+^\uparrow$ [with projection $\pi : F_o(M^4) \rightarrow M^4$] corresponds to the proper subgroup $L_+^\uparrow \subset SO(3,1)$ of the Lorentz group. Therefore, global frames (tetrads) and coframes (cotetrads) exist. A spin structure for $F_o(M^4)$ is, in this case, the trivial spin principal SL(2,C)-bundle $S(M^4) = M^4 \times SL(2,C)$ [with projection $\pi_s : S(M^4) \rightarrow M^4$] and a map $\lambda : S(M^4) \rightarrow F_o(M^4)$ such that $\pi(\lambda(p)) = \pi_s(p) \in M^4$ for all $p \in S(M^4)$ and $\lambda(pA) = \lambda(p)\Lambda(A)$ for all $p \in S(M^4)$, $A \in SL(2,C)$, with $\Lambda : SL(2,C) \rightarrow L_+^\uparrow$ the universal covering homomorphism. Then, Dirac fields are defined as cross sections of a bundle associated with $S(M^4)$ [31]. Since $M^4 = R \times \Sigma$ is time- and space-oriented, the hypersurfaces Σ_τ of simultaneity are necessarily space-oriented and are parallelizable (as every 3-manifold [39]): therefore, global triads and cotriads exist. $F(\Sigma_\tau) = \Sigma_\tau \times SO(3)$ is the trivial orthonormal frame SO(3)-bundle and, since one has $\pi_1(SO(3)) = \pi_1(L_+^\uparrow) = Z_2$ for the first homotopy group, one can define SU(2) spinors on Σ_τ [40,41].

iv) The noncompact parallelizable simultaneity 3-manifolds (the Cauchy surfaces) Σ_τ are assumed to be topologically trivial, geodesically complete [so that the Hopf-Rinow theorem [30] assures metric completeness of the Riemannian 3-manifold $(\Sigma_\tau, {}^3g)$] and, finally, diffeomorphic to R^3. These 3-manifolds have the same manifold structure as Euclidean spaces [30]: a) the geodesic exponential map $Exp_p : T_p\Sigma_\tau \rightarrow \Sigma_\tau$ is a diffeomorphism (Hadamard theorem); b) the sectional curvature is less or equal zero everywhere; c) they have no "conjugate locus" [i.e. there are no pairs of conjugate Jacobi points (intersection points of distinct geodesics through them) on any geodesic] and no "cut locus" [i.e. no closed geodesics through any point]. In these manifolds two points determine a line, so that the "static" tidal forces in Σ_τ due to the 3-curvature tensor are repulsive; instead in M^4 the tidal forces due to the 4-curvature tensor are attractive, since they describe gravitation, which is always attractive, and this implies that the sectional 4-curvature of timelike tangent planes must be negative (this is the source of the singularity theorems) [30]. In 3-manifolds not of this class one has to give a physical (topological) interpretation of "static" quantities like the two quoted loci. In particular, these 3-manifolds have global charts inherited by R^3 through the diffeomorphism. Given a Cauchy surface Σ_{τ_o} of this type and a set of Cauchy data for the gravitational field (and for matter, if present), the Hamiltonian evolution we are going to describe will be valid from τ_o till $\tau_o + \triangle\tau$, where the interval $\triangle\tau$ is determined by the appearance of either conjugate points on $\Sigma_{\tau_o+\triangle\tau}$ or 4-dimensional singularities in M^4 on its slice $\Sigma_{\tau_o+\triangle\tau}$.

v) Like in Yang-Mills case [5], the 3-spin-connection on the orthogonal frame SO(3)-bundle (and therefore triads and cotriads) will have to be restricted to suited weighted Sobolev spaces to avoid Gribov ambiguities. In turn, this implies the absence of isometries of the noncompact Riemannian 3-manifold $(\Sigma_\tau, {}^3g)$ [see for instance the review paper in Ref. [42]]. All the problems of the boundary conditions on lapse and shift functions and on cotriads will be studied in connection with the Poincaré charges in a future paper.

Diffeomorphisms on Σ_τ ($Diff\, \Sigma_\tau$) will be interpreted in the passive way, following Ref. [26], in accord with the Hamiltonian point of view that infinitesimal diffeomorphisms are generated by taking the Poisson bracket with the 1st class supermomentum constraints [passive diffeomorphisms are also named 'pseudodiffeomorphisms'] . The Lagrangian approach based on the Hilbert action, connects general covariance with the invariance of the action under spacetime diffeomorphisms ($Diff\, M^4$) extended to 4-tensors. Therefore, the moduli space (or superspace or space of 4-geometries) is the space $Riem\, M^4/Diff\, M^4$ [43], where $Riem\, M^4$ is the space of Lorentzian 4-metrics; as shown in Refs. [44,45], superspace, in general, is not a manifold [it is a stratified manifold with singularities [46]] due to the existence (in Sobolev spaces) of 4-metrics and 4-geometries with isometries. See Ref. [47] for the study of great diffeomorphisms, which are connected with the existence of disjoint components of the diffeomorphism group [in Ref. [5] there is the analogous discussion of the connection of winding number with the great gauge transformations]. Instead, in the ADM Hamiltonian formulation of metric gravity [32] space diffeomorphisms are replaced by $Diff\, \Sigma_\tau$ [or better by their induced action on 3-tensors generated by the supermomentum constraints], while time diffeomorphisms are distorted to the transformations generated by the superhamiltonian 1st class constraint [48,27,49] and by the momenta conjugate to the lapse and shift functions. In the Lichnerowicz-York conformal approach to canonical reduction [50,51] [see Refs. [42,52,53] for reviews], one defines, in the case of closed 3-manifolds, the conformal superspace as the space of conformal 3-geometries [namely the space of conformal 3-metrics modulo $Diff\, \Sigma_\tau$ or, equivalently, as $Riem\, \Sigma_\tau$ (the space of Riemannian 3-metrics) modulo $Diff\, \Sigma_\tau$ and conformal transformations ${}^3g \mapsto \phi^4\, {}^3g$ ($\phi > 0$)], because in this approach gravitational dynamics is regarded as the time evolution of conformal 3-geometry [the momentum conjugate to the conformal factor ϕ is replaced by York time [51,54], i.e. the trace of the extrinsic curvature of Σ_τ]. However, the gauge transformations generated by the superhamiltonian constraint are poorly understood. Moreover, the Hamiltonian group of gauge transformations of the ADM theory has 8 (and not 4) generators, because, besides the superhamiltonian and supermomentum constraints, there are the four primary first class constraints giving the vanishing of the canonical momenta conjugate to the lapse and shift functions [whose gauge nature is connected with the gauge nature (conventionality) of simultaneity [55] and of the standards of time and length]. A discussion of these problems and of general covariance versus Dirac's observables will be given in Ref. [56] [as also recently noted in Ref. [57] the problem of observables is still open in canonical gravity].

Our approach to tetrad gravity [see Refs. [58–69] for the existing versions of the theory] utilizes the ADM action of metric gravity with the 4-metric expressed in terms of arbitrary cotetrads, which are parametrized in a particular way in terms of Lorentz-boost parameters and cotetrads adapted to Σ_τ [which, in turn, depend on cotriads on Σ_τ and on lapse and shift functions].

At the Hamiltonian level, the Hamiltonian gauge group contains: i) a $R^3 \times SO(3)$ subgroup replacing the usual Lorentz subgroup due to our parametrization which Abelianizes Lorentz boosts; ii) $Diff\, \Sigma_\tau$ in the sense of the pseudodiffeomorphisms generated by the supermomentum constraints; iii) the gauge transformations generated by a superhamiltonian 1st class constraint; iv) the gauge transformations generated by the momenta conjugate to the lapse and shift functions. In the second paper [56] we shall extract Dirac's observables starting from the symplectic action of infinitesimal diffeomorphisms in $Diff\, \Sigma_\tau$, ignoring the problems on the structure in large of the component of $Diff\,\Sigma_\tau$ connected to the identity when a differential structure is posed on it. Although such global properties can be studied in Yang-Mills theory (since the group of gauge transformations is a Hilbert-Lie group), as shown in Ref. [5], and can be applied to the SO(3) gauge transformations of cotriads (in our approach the Lorentz boosts are automatically Abelianized), one has that SO(3) gauge transformations and $Diff\,\Sigma_\tau$ do not commute. Therefore, in tetrad gravity the group of SO(3) gauge transformations is an invariant subgroup of a larger group, the group of automorphisms of the SO(3) frame bundle, containing also $Diff\,\Sigma_\tau$ and again the global situation in the large is of difficult control [$Diff\,\Sigma_\tau$ is an inductive limit of Hilbert-Lie groups [70], but the global properties of its group manifold are not well understood].

In this first paper, after a review of the formalisms needed in this and in the future papers, we shall introduce our parametrization of the cotetrads, we shall give the Lagrangian and Hamiltonian formulations of tetrad gravity and we shall study the algebra of the resulting fourteen first class constraints.

Section II is devoted to a review of 4-dimensional pseudo-Riemannian and 3-dimensional Riemannian manifolds asymptotically flat at spatial infinity, of the tetrad formalism and of the Lagrangians used for general relativity.

In Section III, Σ_τ-adapted tetrads and triads are introduced and the new parametrization of cotetrads is defined.

In Section IV such parametrized cotetrads are inserted in the ADM metric action and the Hamiltonian formulation is performed with the identification of fourteen first class constraints. The comparison with other formulations of tetrad gravity is done.

In Section V there is a comparison with ADM canonical metric gravity and a comment on the Hamiltonian formulation of the harmonic gauge.

In the Conclusions the next step, namely the identification of the Dirac observables with respect to the gauge transfomations generated by thirteen constraints (only the superhamitonian constraint is not treated), is delineated.

In Appendix A relevant 4-tensors are described in Σ_τ-adapted holonomic coordinates and in Appendix B their Hamiltonian expression is given.

II. NOTATIONS.

In this Section we shall introduce the notations needed to define the ADM tetrads and triads of the next Section.

Let M^4 be a torsion-free, globally hyperbolic, asymptotically flat pseudo-Riemannian (or Lorentzian) 4-manifold, whose nondegenerate 4-metric tensor ${}^4g_{\mu\nu}(x)$ has Lorentzian signature $\epsilon(+,-,-,-)$ with $\epsilon = \pm 1$ according to particle physics and general relativity conventions respectively; the inverse 4-metric is ${}^4g^{\mu\nu}(x)$ with ${}^4g^{\mu\rho}(x){}^4g_{\rho\nu}(x) = \delta^\mu_\nu$. We shall denote with Greek letters $\mu, \nu, ..$ $(\mu = 0, 1, 2, 3)$, the world indices and with Greek letters inside round brackets $(\alpha), (\beta), ..$, flat Minkowski indices [with flat 4-metric tensor ${}^4\eta_{(\alpha)(\beta)} = \epsilon(+,-,-,-)$ in Cartesian coordinates]; analogously, $a, b, ..$, and $(a), (b), ..$, [a=1,2,3], will denote world and flat 3-space indices.

We shall follow the conventions of Refs. [71,53] for $\epsilon = -1$ and those of Ref. [72] for $\epsilon = +1$ [i.e. the conventions of standard textbooks; see also Ref. [38] for many results (this book is consistent with Ref. [71], even if its index conventions are different)].

The coordinates of a chart of the atlas of M^4 will be denoted $\{x^\mu\}$. M^4 is assumed to be orientable; its volume element in any right-handed coordinate basis is $-\eta\sqrt{{}^4g}\, d^4x$ [η is a sign connected with the choice of the orientation and ${}^4g = |det\, {}^4g_{\mu\nu}|$; with $\eta = \epsilon$ we get the choice of Ref. [71] for $\epsilon = -1$ and of Ref. [72] for $\epsilon = +1$]. In the coordinate bases $e_\mu = \partial_\mu$ and dx^μ for vector fields [TM^4] and one-forms [or covectors; T^*M^4] respectively, the unique metric-compatible Levi-Civita affine connection has the symmetric Christoffel symbols ${}^4\Gamma^\mu_{\alpha\beta} = {}^4\Gamma^\mu_{\beta\alpha} = \frac{1}{2}{}^4g^{\mu\nu}(\partial_\alpha\, {}^4g_{\beta\nu} + \partial_\beta\, {}^4g_{\alpha\nu} - \partial_\nu\, {}^4g_{\alpha\beta})$ as connection coefficients [${}^4\Gamma^\mu_{\mu\nu} = \partial_\nu\sqrt{{}^4g}$] and the associated covariant derivative is denoted ${}^4\nabla_\mu$ [or with a semicolon ";"]: ${}^4V^\mu{}_{;\nu} = {}^4\nabla_\nu\, {}^4V^\mu = \partial_\nu\, {}^4V^\mu + {}^4\Gamma^\mu_{\nu\alpha}\, {}^4V^\alpha$, with the metric compatibility condition being ${}^4\nabla_\rho\, {}^4g^{\mu\nu} = 0$. The Christoffel symbols are not tensors. If, instead of the chart of M^4 with coordinates $\{x^\mu\}$, we choose another chart of M^4, overlapping with the previous one, with coordinates $\{x^{'\mu} = x^{'\mu}(x)\}$ [$x^{'\mu}(x)$ smooth functions], in the overlap of the two charts we have the following transformation properties under general smooth coordinate transformations or diffeomorphisms of M^4 [$Diff\, M^4$, the gauge group of Einstein-Hilbert Lagrangian] of ${}^4g_{\alpha\beta}(x)$ and of ${}^4\Gamma^\mu_{\alpha\beta}(x)$ respectively

$$
\begin{aligned}
{}^4g^{'}_{\alpha\beta}(x^{'}(x)) &= \frac{\partial x^\mu}{\partial x^{'\alpha}}\frac{\partial x^\nu}{\partial x^{'\beta}}\, {}^4g_{\mu\nu}(x),\\
{}^4\Gamma^{'\mu}_{\alpha\beta}(x^{'}(x)) &= \frac{\partial x^{'\mu}}{\partial x^\nu}\frac{\partial x^\gamma}{\partial x^{'\alpha}}\frac{\partial x^\delta}{\partial x^{'\beta}}\, {}^4\Gamma^\nu_{\gamma\delta}(x) + \frac{\partial^2 x^\nu}{\partial x^{'\alpha}\partial x^{'\beta}}\frac{\partial x^{'\mu}}{\partial x^\nu}.
\end{aligned}
\tag{1}
$$

For a tensor density of weight W, ${}^4\mathcal{T}^{\mu ...}{}_{\alpha ...} = ({}^4g)^{-W/2}\, {}^4T^{\mu ...}{}_{\alpha ...}$, we have ${}^4\mathcal{T}^{\mu ...}{}_{\alpha ..;\rho} = ({}^4g)^{-W/2}[({}^4g)^{W/2}\, {}^4\mathcal{T}^{\mu ...}{}_{\alpha ...}]_{;\rho} = ({}^4g)^{-W/2}\, {}^4T^{\mu ...}{}_{\alpha ..;\rho} = \partial_\rho\, {}^4\mathcal{T}^{\mu ...}{}_{\alpha ...} + {}^4\Gamma^\mu_{\rho\nu}\, {}^4\mathcal{T}^{\nu ...}{}_{\alpha ...} + \cdots - {}^4\Gamma^\beta_{\rho\alpha}\, {}^4\mathcal{T}^{\mu ...}{}_{\beta ...} - \cdots + W\, {}^4\Gamma^\sigma_{\sigma\rho}\, {}^4\mathcal{T}^{\mu ...}{}_{\alpha ...}$ $[\partial_\rho({}^4g)^{-W/2} + W({}^4g)^{-W/2}\, {}^4\Gamma^\mu_{\mu\rho} = 0]$. The covariant divergence of a vector density of weight -1 is equal to its ordinary divergence: ${}^4\nabla_\mu\, {}^4\mathcal{T}^\mu = \partial_\mu\, {}^4\mathcal{T}^\mu + {}^4\Gamma^\mu_{\mu\nu}\, \mathcal{T}^\nu - {}^4\Gamma^\mu_{\mu\nu}\, {}^4\mathcal{T}^\nu = \partial_\mu\, {}^4\mathcal{T}^\mu$. For the Lie derivatives we have: i) $\mathcal{L}_{V^\alpha\partial_\alpha}\, {}^4g_{\mu\nu} = V_{\mu;\nu} + V_{\nu;\mu}$; ii) $\mathcal{L}_{V^\alpha\partial_\alpha}\sqrt{{}^4g} = \frac{1}{2}\sqrt{{}^4g}\, {}^4g^{\mu\nu}\mathcal{L}_{V^\alpha\partial_\alpha}\, {}^4g_{\mu\nu} = \partial_\mu(\sqrt{{}^4g}V^\mu)$ and ${}^4g^{w/2}\mathcal{L}_{V^\alpha\partial_\alpha}\, {}^4g^{-w/2} = -{}^4g^{-w/2}\mathcal{L}_{V^\alpha\partial_\alpha}\, {}^4g^{w/2} = -\frac{w}{\sqrt{{}^4g}}\mathcal{L}_{V^\alpha\partial_\alpha}\sqrt{{}^4g}$; iii) $\mathcal{L}_{V^\alpha\partial_\alpha}\, {}^4\mathcal{T}^\mu = -w\partial_\alpha V^\alpha\, {}^4\mathcal{T}^\mu + V^\alpha\partial_\alpha\, {}^4\mathcal{T}^\mu - \partial_\alpha V^\mu\, {}^4\mathcal{T}^\alpha$ and $\mathcal{L}_{V^\alpha\partial_\alpha}(\sqrt{{}^4g}f) = \partial_\mu(\sqrt{{}^4g}fV^\mu)$.

The Riemann curvature tensor is [this is the definition of Ref. [71] for $\epsilon = -1$; for $\epsilon = +1$ it coincides with minus the definition of Ref. [72]]

$$
{}^4R^\alpha{}_{\mu\beta\nu} = {}^4\Gamma^\alpha_{\beta\rho}\, {}^4\Gamma^\rho_{\nu\mu} - {}^4\Gamma^\alpha_{\nu\rho}\, {}^4\Gamma^\rho_{\beta\mu} + \partial_\beta\, {}^4\Gamma^\alpha_{\mu\nu} - \partial_\nu\, {}^4\Gamma^\alpha_{\beta\mu},
$$

$$
\begin{aligned}
{}^4R_{\alpha\mu\beta\nu} &= {}^4g_{\alpha\gamma}\, {}^4R^\gamma{}_{\mu\beta\nu} = \\
&= \frac{1}{2}(\partial_\alpha\partial_\nu\, {}^4g_{\mu\beta} + \partial_\mu\partial_\beta\, {}^4g_{\alpha\nu} - \partial_\mu\partial_\nu\, {}^4g_{\alpha\beta} - \partial_\alpha\partial_\beta\, {}^4g_{\mu\nu}) + \\
&+ {}^4g_{\rho\sigma}({}^4\Gamma^\rho_{\alpha\nu}\, {}^4\Gamma^\sigma_{\mu\beta} - {}^4\Gamma^\rho_{\alpha\beta}\, {}^4\Gamma^\sigma_{\mu\nu}) = -{}^4R_{\alpha\mu\nu\beta} = -{}^4R_{\mu\alpha\beta\nu} = {}^4R_{\beta\nu\alpha\mu},
\end{aligned} \tag{2}
$$

while the Ricci tensor and the curvature scalar are defined as

$$
\begin{aligned}
{}^4R_{\mu\nu} &= {}^4R_{\nu\mu} = {}^4R^\beta{}_{\mu\beta\nu} = \\
&= \partial_\rho\, {}^4\Gamma^\rho_{\mu\nu} - \partial_\nu\, {}^4\Gamma^\rho_{\rho\mu} + {}^4\Gamma^\rho_{\mu\nu}\, {}^4\Gamma^\beta_{\beta\rho} - {}^4\Gamma^\rho_{\mu\beta}\, {}^4\Gamma^\beta_{\nu\rho} = \\
&= \frac{1}{\sqrt{{}^4g}}\partial_\rho(\sqrt{{}^4g}\, {}^4\Gamma^\rho_{\mu\nu}) - \partial_\mu\partial_\nu\, ln\, \sqrt{{}^4g} - {}^4\Gamma^\rho_{\mu\beta}\, {}^4\Gamma^\beta_{\nu\rho},
\end{aligned}
$$

$$
\begin{aligned}
{}^4R &= {}^4g^{\mu\nu}\, {}^4R_{\mu\nu} = {}^4R^{\mu\nu}{}_{\mu\nu} = \\
&= {}^4g^{\mu\nu}\, {}^4g^{\alpha\beta}[\partial_\alpha\partial_\nu\, {}^4g_{\mu\beta} - \partial_\mu\partial_\nu\, {}^4g_{\alpha\beta} + {}^4g_{\rho\sigma}({}^4\Gamma^\rho_{\alpha\nu}\, {}^4\Gamma^\sigma_{\mu\beta} - {}^4\Gamma^\rho_{\alpha\beta}\, {}^4\Gamma^\sigma_{\mu\nu})] = \\
&= {}^4g^{\mu\nu}({}^4\Gamma^\rho_{\mu\beta}\, {}^4\Gamma^\beta_{\nu\rho} - {}^4\Gamma^\rho_{\mu\nu}\, {}^4\Gamma^\beta_{\beta\rho}) + \frac{1}{\sqrt{{}^4g}}\partial_\rho[\sqrt{{}^4g}({}^4g^{\mu\nu}\, {}^4\Gamma^\rho_{\mu\nu} - {}^4g^{\mu\rho}\, {}^4\Gamma^\nu_{\nu\mu})].
\end{aligned} \tag{3}
$$

The first and second Bianchi identities have the following expression [${}^4G_{\mu\nu}$ is the Einstein tensor and ${}^4\nabla_\mu\, {}^4G^{\mu\nu} \equiv 0$ are the Bianchi identities]

$$
{}^4R^\alpha{}_{\mu\beta\nu} + {}^4R^\alpha{}_{\beta\nu\mu} + {}^4R^\alpha{}_{\nu\mu\beta} \equiv 0,
$$

$$
({}^4\nabla_\gamma\, {}^4R)^\alpha{}_{\mu\beta\nu} + ({}^4\nabla_\beta\, {}^4R)^\alpha{}_{\mu\nu\gamma} + ({}^4\nabla_\nu\, {}^4R)^\alpha{}_{\mu\gamma\beta} \equiv 0,
$$

$$
\begin{aligned}
&\Rightarrow\ ({}^4\nabla_\gamma\, {}^4R^{(ricci)})_{\mu\nu} + ({}^4\nabla_\alpha\, {}^4R)^\alpha{}_{\mu\nu\gamma} - ({}^4\nabla_\nu\, {}^4R^{(ricci)})_{\mu\gamma} \equiv 0, \\
&\Rightarrow\ {}^4\nabla_\mu\, {}^4G^{\mu\nu} \equiv 0, \qquad {}^4G_{\mu\nu} = {}^4R_{\mu\nu} - \frac{1}{2}{}^4g_{\mu\nu}\, {}^4R, \quad {}^4G = -{}^4R.
\end{aligned} \tag{4}
$$

There are 20 independent components of the Riemann tensor in four dimensions due to its symmetry properties.

The Weyl or conformal tensor (which vanish if and only if M^4 is conformally flat) is defined as the completely trace-free part of the Riemann tensor [in empty spacetime Einstein's equations imply ${}^4C_{\alpha\mu\beta\nu} \,\circeq\, {}^4R_{\alpha\mu\beta\nu}$, where $\circeq$ means evaluated on the solution of the equations of motion]

$$
\begin{aligned}
{}^4C_{\alpha\mu\beta\nu} &= {}^4R_{\alpha\mu\beta\nu} + \frac{1}{2}({}^4R_{\alpha\beta}\, {}^4g_{\mu\nu} - {}^4R_{\mu\beta}\, {}^4g_{\alpha\nu} + {}^4R_{\mu\nu}\, {}^4g_{\alpha\beta} - {}^4R_{\alpha\nu}\, {}^4g_{\mu\beta}) + \\
&+ \frac{1}{6}({}^4g_{\alpha\beta}\, {}^4g_{\mu\nu} - {}^4g_{\alpha\nu}\, {}^4g_{\mu\beta})\, {}^4R
\end{aligned} \tag{5}
$$

Let our globally hyperbolic spacetime M^4 be foliated with spacelike Cauchy hypersurfaces Σ_τ, obtained with the embeddings $i_\tau : \Sigma \rightarrow \Sigma_\tau \subset M^4$, $\vec{\sigma} \mapsto x^\mu = z^\mu(\tau, \vec{\sigma})$, of a 3-manifold Σ in M^4 [$\tau : M^4 \rightarrow R$ is a global, timelike, future-oriented function labelling the leaves of the foliation; x^μ are local coordinates in a chart of M^4; $\vec{\sigma} = \{\sigma^r\}$, r=1,2,3, are local coordinates in a chart of Σ, which is diffeomorphic to R^3; we shall use the notation $\sigma^A = (\sigma^\tau = \tau; \vec{\sigma})$, $A = \tau, r$, and $z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma})$]. Let $n^\mu(\sigma)$ and $l^\mu(\sigma) = N(\sigma) n^\mu(\sigma)$ be the controvariant timelike normal and unit normal $[{}^4g_{\mu\nu}(z(\sigma)) l^\mu(\sigma) l^\nu(\sigma) = \epsilon]$ to Σ_τ at the point $z(\sigma) \in \Sigma_\tau$. The positive function $N(\sigma) > 0$ is the lapse function: $N(\sigma) d\tau$ measures the proper time interval at $z(\sigma) \in \Sigma_\tau$ between Σ_τ and $\Sigma_{\tau+d\tau}$. The shift functions $N^r(\sigma)$ are defined so that $N^r(\sigma) d\tau$ describes the horizontal shift on Σ_τ such that, if $z^\mu(\tau + d\tau, \vec{\sigma} + d\vec{\sigma}) \in \Sigma_{\tau+d\tau}$, then $z^\mu(\tau + d\tau, \vec{\sigma} + d\vec{\sigma}) \approx z^\mu(\tau, \vec{\sigma}) + N(\tau, \vec{\sigma}) d\tau l^\mu(\tau, \vec{\sigma}) + [d\sigma^r + N^r(\tau, \vec{\sigma}) d\tau] \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^r}$; therefore, we have $\frac{\partial z^\mu(\sigma)}{\partial \tau} = N(\sigma) l^\mu(\sigma) + N^r(\sigma) \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^r}$ for the so called evolution vector. For the covariant unit normal to Σ_τ we have $l_\mu(\sigma) = {}^4g_{\mu\nu}(z(\sigma)) l^\nu(\sigma) = N(\sigma) \partial_\mu \tau|_{x=z(\sigma)}$.

Instead of local coordinates x^μ for M^4, we use local coordinates σ^A on $R \times \Sigma \approx M^4$ $[x^\mu = z^\mu(\sigma)$ with inverse $\sigma^A = \sigma^A(x)]$, i.e. a Σ_τ-adapted holonomic coordinate basis for vector fields $\partial_A = \frac{\partial}{\partial \sigma^A} \in T(R \times \Sigma) \mapsto b^\mu_A(\sigma) \partial_\mu = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} \partial_\mu \in TM^4$, and for differential one-forms $dx^\mu \in T^*M^4 \mapsto d\sigma^A = b^A_\mu(\sigma) dx^\mu = \frac{\partial \sigma^A(z)}{\partial z^\mu} dx^\mu \in T^*(R \times \Sigma)$. Let us note that in the flat Minkowski spacetime the transformation coefficients $b^A_\mu(\sigma)$ and $b^\mu_A(\sigma)$ become the flat orthonormal cotetrads $z^A_\mu(\sigma) = \frac{\partial \sigma^A(x)}{\partial x^\mu}|_{x=z(\sigma)}$ and tetrads $z^\mu_A(\sigma) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A}$ of Ref. [11]. The induced 4-metric and inverse 4-metric become in the new basis

$$
\begin{aligned}
{}^4g(x) &= {}^4g_{\mu\nu}(x) dx^\mu \otimes dx^\nu = {}^4g_{AB}(z(\sigma)) d\sigma^A \otimes d\sigma^B, \\
\\
{}^4g_{\mu\nu} &= b^A_\mu \, {}^4g_{AB} b^B_\nu = \\
&= \epsilon \, (N^2 - {}^3g_{rs} N^r N^s) \partial_\mu \tau \partial_\nu \tau - \epsilon \, {}^3g_{rs} N^s (\partial_\mu \tau \partial_\nu \sigma^r + \partial_\nu \tau \partial_\mu \sigma^r) - \epsilon \, {}^3g_{rs} \partial_\mu \sigma^r \partial_\nu \sigma^s = \\
&= \epsilon \, l_\mu l_\nu - \epsilon \, {}^3g_{rs} (\partial_\mu \sigma^r + N^r \, \partial_\mu \tau)(\partial_\nu \sigma^s + N^s \, \partial_\nu \tau), \\
\\
&\Rightarrow {}^4g_{AB} = \{ {}^4g_{\tau\tau} = \epsilon (N^2 - {}^3g_{rs} N^r N^s); {}^4g_{\tau r} = -\epsilon \, {}^3g_{rs} N^s; {}^4g_{rs} = -\epsilon \, {}^3g_{rs} \} = \\
&= \epsilon [l_A l_B - {}^3g_{rs} (\delta^r_A + N^r \delta^\tau_A)(\delta^s_B + N^s \delta^\tau_B)], \\
\\
{}^4g^{\mu\nu} &= b^\mu_A \, {}^4g^{AB} b^\nu_B = \\
&= \frac{\epsilon}{N^2} \partial_\tau z^\mu \partial_\tau z^\nu - \frac{\epsilon \, N^r}{N^2} (\partial_\tau z^\mu \partial_r z^\nu + \partial_\tau z^\nu \partial_r z^\mu) - \epsilon ({}^3g^{rs} - \frac{N^r N^s}{N^2}) \partial_r z^\mu \partial_s z^\nu = \\
&= \epsilon [\, l^\mu l^\nu - \, {}^3g^{rs} \partial_r z^\mu \partial_s z^\nu], \\
\\
&\Rightarrow {}^4g^{AB} = \{ {}^4g^{\tau\tau} = \frac{\epsilon}{N^2}; {}^4g^{\tau r} = -\frac{\epsilon \, N^r}{N^2}; {}^4g^{rs} = -\epsilon ({}^3g^{rs} - \frac{N^r N^s}{N^2}) \} = \\
&= \epsilon [l^A l^B - {}^3g^{rs} \delta^A_r \delta^B_s], \\
\\
l^A &= l^\mu b^A_\mu = N \, {}^4g^{A\tau} = \frac{\epsilon}{N} (1; -N^r), \\
l_A &= l_\mu b^\mu_A = N \partial_A \tau = N \delta^\tau_A = (N; \vec{0}).
\end{aligned}
\tag{6}
$$

Here, we introduced the 3-metric of Σ_τ: ${}^3g_{rs} = -\epsilon \, {}^4g_{rs}$ with signature (+++). If ${}^4\gamma^{rs}$ is the inverse of the spatial part of the 4-metric $[{}^4\gamma^{ru} \, {}^4g_{us} = \delta^r_s]$, the inverse of the 3-metric is ${}^3g^{rs} = -\epsilon \, {}^4\gamma^{rs}$ $[{}^3g^{ru} \, {}^3g_{us} = \delta^r_s]$. ${}^3g_{rs}(\tau, \vec{\sigma})$ are the components of the "first fundamental form" of the Riemann 3-manifold $(\Sigma_\tau, {}^3g)$ and we have

$$ds^2 = {}^4g_{\mu\nu}dx^\mu dx^\nu = \epsilon(N^2 - {}^3g_{rs}N^rN^s)(d\tau)^2 - 2\epsilon\, {}^3g_{rs}N^s d\tau d\sigma^r - \epsilon\, {}^3g_{rs}d\sigma^r d\sigma^s = \epsilon\Big[N^2(d\tau)^2 - {}^3g_{rs}(d\sigma^r + N^r d\tau)(d\sigma^s + N^s d\tau)\Big]$$

for the line element in M^4. We must have $\epsilon\, {}^4g_{oo} > 0$, $\epsilon\, {}^4g_{ij} < 0$, $\begin{vmatrix} {}^4g_{ii} & {}^4g_{ij} \\ {}^4g_{ji} & {}^4g_{jj} \end{vmatrix} > 0$, $\epsilon\, det\, {}^4g_{ij} > 0$.

If we define $g = {}^4g = |\, det\, ({}^4g_{\mu\nu})\, |$ and $\gamma = {}^3g = |\, det\, ({}^3g_{rs})\, |$, we also have

$$\begin{aligned} N &= \sqrt{\frac{{}^4g}{{}^3g}} = \frac{1}{\sqrt{{}^4g^{\tau\tau}}} = \sqrt{\frac{g}{\gamma}} = \sqrt{{}^4g_{\tau\tau} - \epsilon\, {}^3g^{rs}\, {}^4g_{\tau r}{}^4g_{\tau s}}, \\ N^r &= -\epsilon\, {}^3g^{rs}\, {}^4g_{\tau s} = -\frac{{}^4g^{\tau r}}{{}^4g^{\tau\tau}}, \quad N_r = {}^3g_{rs}N^s = -\epsilon\; {}^4g_{rs}N^s = -\epsilon {}^4g_{\tau r}. \end{aligned} \tag{7}$$

Let us remark [see Ref. [24]] that in the study of space and time measurements the equation $ds^2 = 0$ [use of light signals for the synchronization of clocks] and the definition $d\bar{\tau} = \sqrt{\epsilon\, {}^4g_{oo}}dx^o$ of proper time [$\sqrt{\epsilon\, {}^4g_{oo}}$ determines the ratio between the rates of a standard clock at rest and a coordinate clock at the same point] imply the use in M^4 of a 3-metric ${}^3\tilde{\gamma}_{rs} = {}^4g_{rs} - \frac{{}^4g_{or}\, {}^4g_{os}}{{}^4g_{oo}} = -\epsilon({}^3g_{rs} + \frac{N_rN_s}{\epsilon\, {}^4g_{oo}})$ with the covariant shift functions $N_r = {}^3g_{rs}N^s = -\epsilon\, {}^4g_{or}$, which are connected with the conventionality of simultaneity [55] and with the direction dependence of the velocity of light [$c(\vec{n}) = \sqrt{\epsilon\, {}^4g_{oo}}/(1 + N_rn^r)$ in direction $\vec{n}$].

See Refs. [73,71,27] for the 3+1 decomposition of 4-tensors on M^4. The horizontal projector ${}^3h_\mu^\nu = \delta_\mu^\nu - \epsilon\, l_\mu l^\nu$ on Σ_τ defines the 3-tensor fields on Σ_τ starting from the 4-tensor fields on M^4.

In the standard (not Hamiltonian) description of the 3+1 decomposition we utilize a Σ_τ-adapted nonholonomic noncoordinate basis [$\bar{A} = (l; r)$]

$$\begin{aligned} \hat{b}^\mu_{\bar{A}}(\sigma) &= \{\hat{b}^\mu_l(\sigma) = \epsilon l^\mu(\sigma) = N^{-1}(\sigma)[b^\mu_\tau(\sigma) - N^r(\sigma)b^\mu_r(\sigma)]; \\ &\qquad \hat{b}^\mu_r(\sigma) = b^\mu_r(\sigma)\}, \\ \hat{b}^{\bar{A}}_\mu(\sigma) &= \{\hat{b}^l_\mu(\sigma) = l_\mu(\sigma) = N(\sigma)b^\tau_\mu(\sigma) = N(\sigma)\partial_\mu\tau(z(\sigma)); \\ &\qquad \hat{b}^r_\mu(\sigma) = b^r_\mu(\sigma) + N^r(\sigma)b^\tau_\mu(\sigma)\}, \end{aligned}$$

$$\begin{aligned} &\hat{b}^{\bar{A}}_\mu(\sigma)\hat{b}^\nu_{\bar{A}}(\sigma) = \delta^\nu_\mu, \quad \hat{b}^{\bar{A}}_\mu(\sigma)\hat{b}^\mu_{\bar{B}}(\sigma) = \delta^{\bar{A}}_{\bar{B}}, \\ {}^4\bar{g}_{\bar{A}\bar{B}}(z(\sigma)) &= \hat{b}^\mu_{\bar{A}}(\sigma){}^4g_{\mu\nu}(z(\sigma))\hat{b}^\nu_{\bar{B}}(\sigma) = \\ &= \{{}^4\bar{g}_{ll}(\sigma) = \epsilon; {}^4\bar{g}_{lr}(\sigma) = 0; {}^4\bar{g}_{rs}(\sigma) = {}^4g_{rs}(\sigma) = -\epsilon\, {}^3g_{rs}\}, \\ {}^4\bar{g}^{\bar{A}\bar{B}} &= \{{}^4\bar{g}^{ll} = \epsilon; {}^4\bar{g}^{lr} = 0; {}^4\bar{g}^{rs} = {}^4\gamma^{rs} = -\epsilon{}^3g^{rs}\}, \end{aligned}$$

$$\begin{aligned} X_{\bar{A}} &= \hat{b}^\mu_{\bar{A}}\partial_\mu = \{X_l = \frac{1}{N}(\partial_\tau - N^r\partial_r); \partial_r\}, \\ \theta^{\bar{A}} &= \hat{b}^{\bar{A}}_\mu dx^\mu = \{\theta^l = Nd\tau; \theta^r = d\sigma^r + N^r d\tau\}, \end{aligned}$$

$$\Rightarrow l_\mu(\sigma)b^\mu_r(\sigma) = 0, \quad l^\mu(\sigma)b^r_\mu(\sigma) = -N^r(\sigma)/N(\sigma),$$

$$
\begin{aligned}
&l^{\bar{A}} = l^\mu \hat{b}^{\bar{A}}_\mu = (\epsilon; l^r + N^r l^\tau) = (\epsilon; \vec{0}),\\
&l_{\bar{A}} = l_\mu \hat{b}^\mu_{\bar{A}} = (1; l_r) = (1; \vec{0}).
\end{aligned}
\tag{8}
$$

We have ${}^3h_{\mu\nu} = {}^4g_{\mu\nu} - \epsilon l_\mu l_\nu = -\epsilon\, {}^3g_{rs}(b^r_\mu + N^r b^\tau_\mu)(b^s_\mu + N^s b^\tau_\mu) = -\epsilon\, {}^3g_{rs}\hat{b}^r_\mu \hat{b}^s_\nu$ and for a 4-vector ${}^4V^\mu = {}^4V^{\bar{A}}\hat{b}^\mu_{\bar{A}} = {}^4V^l l^\mu + {}^4V^r \hat{b}^\mu_r$ we have ${}^3V^\mu = {}^3V^r \hat{b}^\mu_r = {}^3h^\mu_\nu\, {}^4V^\nu$, ${}^3V^r = \hat{b}^r_\mu\, {}^3V^\mu$.

The nonholonomic basis in Σ_τ-adapted coordinates is

$$
\begin{aligned}
&\hat{b}^{\bar{A}}_A = \hat{b}^{\bar{A}}_\mu b^\mu_A = \{\hat{b}^l_A = l_A;\ \hat{b}^r_A = \delta^r_A + N^r \delta^\tau_A\}\\
&\hat{b}^A_{\bar{A}} = \hat{b}^\mu_{\bar{A}} b^A_\mu = \{\hat{b}^A_l = \epsilon l^A;\ \hat{b}^A_r = \delta^A_r\}.
\end{aligned}
$$

One can show the following results concerning the Lie derivative along the unit normal: i) $\mathcal{L}_l\, \hat{b}^\mu_r = -\mathcal{L}_l\, l^\mu = N^{-1}(\partial_r N l^\mu + \partial_r N^s \hat{b}^\mu_s) = l^\nu \hat{b}^\mu_{r;\nu} - \hat{b}^\nu_r l^\mu{}_{;\nu}$; ii) $\mathcal{L}_l \hat{b}^r_\mu = -n^{-1}\partial_s N^r \hat{b}^s_\mu$.

The 3-dimensional covariant derivative [denoted ${}^3\nabla$ or with the subscript "$|$"] of a 3-dimensional tensor ${}^3T^{\mu_1..\mu_p}{}_{\nu_1..\nu_q}$ of rank (p,q) is the 3-dimensional tensor of rank (p,q+1) ${}^3\nabla_\rho\, {}^3T^{\mu_1..\mu_p}{}_{\nu_1..\nu_q} = {}^3T^{\mu_1..\mu_p}{}_{\nu_1..\nu_q|\rho} = {}^3h^{\mu_1}_{\alpha_1}\cdots {}^3h^{\mu_p}_{\alpha_p}\, {}^3h^{\beta_1}_{\nu_1}\cdots {}^3h^{\beta_q}_{\nu_q}\, {}^3h^\sigma_\rho\, {}^4\nabla_\sigma\, {}^3T^{\alpha_1..\alpha_p}{}_{\beta_1..\beta_q}$. For (1,0) and (0,1) tensors we have: ${}^3\nabla_\rho\, {}^3V^\mu = {}^3V^\mu{}_{|\rho} = {}^3V^r{}_{|s}\, \hat{b}^\mu_r \hat{b}^s_\rho$, ${}^3\nabla_s\, {}^3V^r = {}^3V^r{}_{|s} = \partial_s\, {}^3V^r + {}^3\Gamma^r_{su}\, {}^3V^u$ and ${}^3\nabla_\rho\, {}^3\omega_\mu = {}^3\omega_{\mu|\rho} = {}^3\omega_{r|s}\, \hat{b}^r_\mu \hat{b}^s_\rho$, ${}^3\nabla_s\, {}^3\omega_r = {}^3\omega_{r|s} = \partial_s\, {}^3\omega_r - {}^3\Gamma^u_{rs}\, {}^3\omega_u$ respectively.

The 3-dimensional Christoffel symbols are ${}^3\Gamma^u_{rs} = \hat{b}^u_\mu [{}^3\nabla_\rho\, \hat{b}^\mu_r]\hat{b}^\rho_s = \hat{b}^u_\mu \hat{b}^\mu_{r|\rho} \hat{b}^\rho_s = \frac{1}{2}{}^3g^{uv}(\partial_s\, {}^3g_{vr} + \partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs})$ and the metric compatibility [Levi-Civita connection on the Riemann 3-manifold $(\Sigma_\tau, {}^3g)$] is ${}^3\nabla_\rho\, {}^3g_{\mu\nu} = {}^3g_{\mu\nu|\rho} = 0$ [${}^3g_{\mu\nu} = -\epsilon\, {}^3h_{\mu\nu} = {}^3g_{rs}\hat{b}^r_\mu \hat{b}^s_\nu$, so that ${}^3\bar{g}_{\bar{A}\bar{B}} = \{{}^3\bar{g}_{ll} = 0; {}^3\bar{g}_{lr} = 0; {}^3\bar{g}_{rs} = -\epsilon\, {}^3g_{rs}\}$]. It is then possible to define parallel transport on Σ_τ. The 3-dimensional curvature Riemann tensor is

$$
\begin{aligned}
&{}^3R^\mu{}_{\alpha\nu\beta}\, {}^3V^\alpha = {}^3V^\alpha{}_{|\beta|\nu} - {}^3V^\alpha{}_{|\nu|\beta},\\
&\Rightarrow {}^3R^r{}_{suv} = \partial_u\, {}^3\Gamma^r_{sv} - \partial_v\, {}^3\Gamma^r_{su} + {}^3\Gamma^r_{uw}\, {}^3\Gamma^w_{sv} - {}^3\Gamma^r_{vw}\, {}^3\Gamma^w_{su}.
\end{aligned}
\tag{9}
$$

For 3-manifolds, the Riemann tensor has only 6 independent components since the Weyl tensor vanishes: this gives the relation ${}^3R_{\alpha\mu\beta\nu} = \frac{1}{2}({}^3R_{\mu\beta}\, {}^3g_{\alpha\nu} + {}^3R_{\alpha\nu}\, {}^3g_{\mu\beta} - {}^3R_{\alpha\beta}\, {}^3g_{\mu\nu} - {}^3R_{\mu\nu}\, {}^3g_{\alpha\beta}) - \frac{1}{6}({}^3g_{\alpha\beta}\, {}^3g_{\mu\nu} - {}^3g_{\alpha\nu}\, {}^3g_{\beta\mu})\, {}^3R$, which expresses the Riemann tensor in terms of the Ricci tensor. A 3-manifold M^3 is conformally flat if and only if its Weyl-Schouten tensor

$$
{}^3C_{\lambda\mu\nu} = {}^3\nabla_\nu\, {}^3R_{\lambda\mu} - {}^3\nabla_\mu\, {}^3R_{\lambda\nu} - \frac{1}{4}({}^3g_{\lambda\mu}\, \partial_\nu\, {}^3R - {}^3g_{\lambda\nu}\, \partial_\mu\, {}^3R)
$$

vanishes [29]. Equivalently one uses the Cotton-York tensor

$$
{}^3\mathcal{H}_{\mu\nu} = \frac{1}{2}\gamma^{1/3}\ {}^3\epsilon_{\alpha\beta\mu}{}^3\nabla^\alpha\, {}^3R^\beta{}_\nu + {}^3\epsilon_{\alpha\beta\nu}{}^3\nabla^\alpha\, {}^3R^\beta{}_\mu,
$$

which satisfies ${}^3g^{\mu\nu}\, {}^3\mathcal{H}_{\mu\nu} = {}^3\nabla^\mu\, {}^3\mathcal{H}_{\mu\nu} = 0$ [71].

The components of the "second fundamental form" of $(\Sigma_\tau, {}^3g)$ is the extrinsic curvature

$$
{}^3K_{\mu\nu} = {}^3K_{\nu\mu} = -\frac{1}{2}\mathcal{L}_l\, {}^3g_{\mu\nu};
$$

one has ${}^4\nabla_\rho\, l^\mu = \epsilon\, {}^3a^\mu l_\rho - {}^3K_\rho{}^\mu$, with the acceleration ${}^3a^\mu = {}^3a^r \hat{b}^\mu_r$ of the observers travelling along the congruence of timelike curves with tangent vector l^μ given by ${}^3a_r = \partial_r \ln N$. On Σ_τ we have

$${}^3K_{rs} = {}^3K_{sr} = \frac{1}{2N}(N_{r|s} + N_{s|r} - \frac{\partial\, {}^3g_{rs}}{\partial\tau}).$$

Moreover, one has:
i) $\hat{b}^\mu_{r;\nu} = \epsilon\, {}^3a_r l^\mu l_\nu - {}^3K_{rs} l^\mu \hat{b}^s_\nu + \epsilon({}^3K_r{}^s - N^{-1}\partial_r N^s) + {}^3\Gamma^u_{rs}\hat{b}^\mu_u \hat{b}^s_\nu$;
ii) $\hat{b}^r_{\mu;\nu} = -\, {}^3a^r l_\mu l_\nu + \epsilon\, {}^3K_s{}^r \hat{b}^s_\nu l_\mu - \epsilon({}^3K_s{}^r - N^{-1}\partial_s N^r)\hat{b}^s_\mu l_\nu - {}^3\Gamma^r_{su}\hat{b}^s_\mu \hat{b}^u_\nu$;
iii) ${}^3a_{\mu|\nu} = {}^3a_{\nu|\mu} = {}^3a_{r|s}\hat{b}^r_\mu \hat{b}^s_\nu = [\partial_r\partial_s \ln N - {}^3\Gamma^u_{rs}\partial_u \ln N]\hat{b}^r_\mu \hat{b}^s_\nu$;
iv) ${}^3a^\mu{}_{;\mu} = {}^3a^\mu{}_{|\mu} + {}^3a^\mu\, {}^3a_\mu$;
v) $l^\mu\, {}^3K_{;\mu} = -(l^\mu\, {}^3K)_{;\mu} + {}^3K^2$;
vi) $\mathcal{L}_l\, {}^3g^{\mu\nu} = l^\mu\, {}^3a^\nu + l^\nu\, {}^3a^\mu + 2\, {}^3K^{\mu\nu}$.

The information contained in the 20 independent components ${}^4R^\mu{}_{\nu\alpha\beta}$ of the curvature Riemann tensor of M^4 is given by the following three projections [see Ref. [74] for the geometry of embeddings; one has ${}^4\bar{R}^r{}_{suv} = {}^3\bar{R}^r{}_{suv}$]

$$\begin{aligned}
&{}^3h^\mu_\rho\, {}^3h^\sigma_\nu\, {}^3h^\gamma_\alpha\, {}^3h^\delta_\beta\, {}^4R^\rho{}_{\sigma\gamma\delta} = {}^4\bar{R}^r{}_{suv}\hat{b}^\mu_r \hat{b}^s_\nu \hat{b}^u_\alpha \hat{b}^v_\beta = {}^3R^\mu{}_{\nu\alpha\beta} + {}^3K_\alpha{}^\mu\, {}^3K_{\beta\nu} - {}^3K_\beta{}^\mu\, {}^3K_{\alpha\nu},\\
&\quad GAUSS\, EQUATION,\\
&\epsilon l_\rho\, {}^3h^\sigma_\nu\, {}^3h^\gamma_\alpha\, {}^3h^\delta_\beta\, {}^4R^\rho{}_{\sigma\gamma\delta} = {}^4\bar{R}^l{}_{suv}\hat{b}^s_\nu \hat{b}^u_\alpha \hat{b}^v_\beta = {}^3K_{\alpha\nu|\beta} - {}^3K_{\beta\nu|\alpha},\\
&\quad CODAZZI - MAINARDI\, EQUATION,\\
&{}^4R_{\mu\sigma\gamma\delta}\, l^\sigma\, l^\gamma\, {}^3h^\delta_\nu = {}^4\bar{R}_{\mu llu}\hat{b}^u_\nu = \epsilon(\mathcal{L}_l\, {}^3K_{\mu\nu} + {}^3K_\mu{}^\rho\, {}^3K_{\rho\nu} + {}^3a_{\mu|\nu} + {}^3a_\mu\, {}^3a_\nu),\\
&\quad RICCI\, EQUATION,\\
&\quad with \qquad \mathcal{L}_l\, {}^3K_{\mu\nu} = l^\alpha\, {}^3K_{\mu\nu;\alpha} - 2\, {}^3K_\mu{}^\alpha\, {}^3K_{\alpha\nu} + 2\epsilon\, {}^3a^\alpha\, {}^3K_{\alpha(\nu}\, l_{\mu)}.
\end{aligned} \tag{10}$$

In the nonholonomic basis we have:

${}^4R^u{}_{rst} = {}^3R^u{}_{rst} + {}^3K_{rs}\, {}^3K_t{}^u - {}^3K_{rt}\, {}^3K_s{}^u$,
${}^4R = {}^3R + {}^3K_{rs}\, {}^3K^{rs} - ({}^3K)^2$,
${}^4\bar{R}^l{}_{rst} = {}^3K_{rt|s} - {}^3K_{rs|t}$,
${}^4\bar{R}^u{}_{lrl} = {}^3a^u{}_{|r} - {}^3a^u\, {}^3a_r + \mathcal{L}_l\, {}^3K_r{}^u - {}^3K_r{}^s\, {}^3K_s{}^u$,
${}^4\bar{R}^u{}_{lrs} = -{}^3g^{ut}\, {}^4\bar{R}^l_{trs}$,
${}^4\bar{R}^l{}_{rls} = {}^3g_{ru}\, {}^4\bar{R}^u{}_{lsl}$.

Then, we can express ${}^4R_{\mu\nu} = \epsilon\, {}^4\bar{R}_{ll} l_\mu l_\nu + \epsilon\, {}^4\bar{R}_{lr}(l_\mu \hat{b}^r_\nu + l_\nu \hat{b}^r_\mu) + {}^4\bar{R}_{rs}\hat{b}^r_\mu \hat{b}^s_\nu$, ${}^4R$ and the Einstein tensor ${}^4G_{\mu\nu} = {}^4R_{\mu\nu} - \frac{1}{2}\, {}^4g_{\mu\nu}\, {}^4R = \epsilon\, {}^4\bar{G}_{ll} l_\mu l_\nu + \epsilon\, {}^4\bar{G}_{lr}(l_\mu \hat{b}^r_\nu + l_\nu \hat{b}^r_\mu) + {}^4\bar{G}_{rs}\hat{b}^r_\mu \hat{b}^s_\nu$ in the nonholonomic basis, with the result:
${}^4\bar{R}_{ll} = {}^3K_{\mu\nu}\, {}^3K^{\mu\nu} - {}^3K^2 + ({}^3a^\mu - {}^3K l^\mu)_{;\mu}$,
${}^4\bar{R}_{lr} = \epsilon({}^3K_r{}^s - \delta^s_r\, {}^3K)_{|s}$,
${}^4\bar{R}_{rs} = -(\mathcal{L}_l\, {}^3K_{rs} - {}^3R_{rs} - {}^3K\, {}^3K_{rs} + 2\, {}^3K_r{}^u\, {}^3K_{us} + {}^3a_{r|s} + {}^3a_r\, {}^3a_s)$,
${}^4R = -\epsilon({}^3R + {}^3K_{rs}\, {}^3K^{rs} - {}^3K^2) - 2\epsilon({}^3a^\mu - {}^3K l^\mu)_{;\mu}$,
${}^4\bar{G}_{ll} = \frac{1}{2}({}^3R + {}^3K^2 - {}^3K_{rs}\, {}^3K^{rs})$,
${}^4\bar{G}_{lr} = \epsilon({}^3K_r{}^s - \delta^s_r\, {}^3K)_{|s}$,

${}^4\bar{G}_{rs} = -\frac{1}{\sqrt{\gamma}}\mathcal{L}_l[\sqrt{\gamma}({}^3K_{rs} - {}^3g_{rs}\,{}^3K)] + {}^3R_{rs} - \frac{1}{2}\,{}^3g_{rs}\,{}^3R + 2({}^3K\,{}^3K_{rs} - {}^3K_r{}^u\,{}^3K_{us}) + \frac{1}{2}\,{}^3g_{rs}({}^3K^2 - {}^3K_{uv}\,{}^3K^{uv}) + N_{|r|s} - {}^3g_{rs}N^{|u}{}_{|u}$.

The Bianchi identities ${}^4G^{\mu\nu}{}_{;\nu} \equiv 0$ imply the following four contracted Bianchi identities [according to which only two of the six equations ${}^4\bar{G}_{rs} \overset{\circ}{=} 0$ are independent]:

$$\frac{1}{N}\partial_\tau\,{}^4\bar{G}_{ll} - \frac{N^r}{N}\partial_r\,{}^4\bar{G}_{ll} - {}^3K\,{}^4\bar{G}_{ll} + \partial_r\,{}^4\bar{G}_l{}^r + (2\,{}^3a_r + {}^3\Gamma^s_{sr})\,{}^4\bar{G}_l{}^r - {}^3K_{rs}\,{}^4\bar{G}^{rs} \equiv 0,$$
$$\frac{1}{N}\partial_\tau\,{}^4\bar{G}_l{}^r - \frac{N^s}{N}\partial_s\,{}^4\bar{G}_l{}^r + {}^3a^r\,{}^4\bar{G}_{ll} - (2\,{}^3K^r{}_s + \delta^r_s\,{}^3K + \frac{\partial_s N^r}{N})\,{}^4\bar{G}_l{}^s + \partial_s\,{}^4\bar{G}^{rs} + ({}^3a_s + {}^3\Gamma^u_{us})\,{}^4\bar{G}^{rs} \equiv 0.$$

The vanishing of ${}^4\bar{G}_{ll}$, ${}^4\bar{G}_{lr}$, corresponds to the four secondary constraints (restrictions of Cauchy data) of the ADM Hamiltonian formalism (see Section V). The four contracted Bianchi identities, ${}^4G^{\mu\nu}{}_{;\nu} \equiv 0$, imply [38] that, if the restrictions of Cauchy data are satisfied initially and the spatial equations ${}^4G_{ij} \overset{\circ}{=} 0$ are satisfied everywhere, then the secondary constraints are satisfied also at later times [see Ref. [42,38] for the initial value problem]. The four contracted Bianchi identities plus the four secondary constraints imply that only two combinations of the Einstein equations contain the accelerations (second time derivatives) of the two (non tensorial) independent degrees of freedom of the gravitational field and that these equations can be put in normal form [this was one of the motivations behind the discovery of the Shanmugadhasan canonical transformations [3]].

The “intrinsic geometry” of Σ_τ is defined by the Riemannian metric ${}^3g_{rs}$ [it allows to evaluate the length of space curves], the Levi-Civita affine connection, i.e. the Christoffel symbols ${}^3\Gamma^u_{rs}$, [for the parallel transport of 3-dimensional tensors on Σ_τ] and the curvature Riemann tensor ${}^3R^r{}_{stu}$ [for the evaluation of the holonomy and for the geodesic deviation equation]. The “extrinsic geometry” of Σ_τ is defined by the lapse N and shift N^r fields [which describe the “evolution” of Σ_τ in M^4] and by the “extrinsic curvature” ${}^3K_{rs}$ [it is needed to evaluate how much a 3-dimensional vector goes outside Σ_τ under spacetime parallel transport and to rebuild the spacetime curvature from the 3-dimensional one].

Besides the local dual coordinate bases ${}^4e_\mu = \partial_\mu$ and dx^μ for TM^4 and T^*M^4 respectively, we can introduce special ‘noncoordinate’ bases ${}^4\hat{E}_{(\alpha)} = {}^4\hat{E}^\mu_{(\alpha)}(x)\partial_\mu$ and its dual ${}^4\hat{\theta}^{(\alpha)} = {}^4\hat{E}^{(\alpha)}_\mu(x)dx^\mu$ [$i_{{}^4\hat{E}_{(\alpha)}}\,{}^4\hat{\theta}^{(\beta)} = {}^4E^{(\alpha)}_\mu\,{}^4E^\mu_{(\beta)} = \delta^{(\beta)}_{(\alpha)} \Rightarrow {}^4\eta_{(\alpha)(\beta)} = {}^4E^\mu_{(\alpha)}\,{}^4g_{\mu\nu}\,{}^4E^\nu_{(\beta)}$; $(\alpha) = (0),(1),(2),(3)$ are numerical indices] with the “vierbeins or tetrads or (local) frames” ${}^4\hat{E}^\mu_{(\alpha)}(x)$, which are, for each point $x^\mu \in M^4$, the matrix elements of matrices $\{{}^4\hat{E}^\mu_{(\alpha)}\} \in GL(4,R)$; the set of one-forms ${}^4\hat{\theta}^{(\alpha)}$ (with ${}^4\hat{E}^{(\alpha)}_\mu(x)$ being the dual “cotetrads”) is also called “canonical” or “soldering” one-form or “coframe”. Since a “frame” ${}^4\hat{E}$ at the point $x^\mu \in M^4$ is a linear isomorphism [31] ${}^4\hat{E} : R^4 \rightarrow T_xM^4$, $\partial_\alpha \mapsto {}^4\hat{E}(\partial_\alpha) = {}^4\hat{E}_{(\alpha)}$, a frame determines a basis ${}^4\hat{E}_{(\alpha)}$ of T_xM^4 [the coframes ${}^4\hat{\theta}$ determine a basis ${}^4\hat{\theta}^{(\alpha)}$ of $T^*_xM^4$] and we can define a principal fiber bundle with structure group GL(4,R), $\pi : L(M^4) \rightarrow M^4$ called the “frame bundle” of M^4 [its fibers are the sets of all the frames over the points $x^\mu \in M^4$; it is an affine bundle, i.e. there is no (global when it exists) cross section playing the role of the identity cross section of vector bundles]; if $\Lambda \in GL(4,R)$, then the free right action of GL(4,R) on $L(M^4)$ is denoted $R_\Lambda({}^4\hat{E}) = {}^4\hat{E} \circ \Lambda$, ${}^4\hat{E}_{(\alpha)} \mapsto {}^4\hat{E}_{(\beta)}\,(\Lambda^{-1})^{(\beta)}{}_{(\alpha)}$. When M^4 is “parallelizable” [i.e. M^4 admits four vector fields which are independent in each point, so that the tangent bundle $T(M^4)$ is trivial, $T(M^4) = M^4 \times R^4$; this is not possible (no hair theorem) for any compact manifold except a torus], as we shall assume, then $L(M^4) = M^4 \times GL(4,R)$ is a trivial principal bundle [i.e. it admits a global cross section $\sigma : M^4 \rightarrow L(M^4)$, $x^\mu \mapsto {}^4_\sigma\hat{E}_{(\alpha)}(x)$]. See Ref. [31] for the differential structure on $L(M^4)$. With the assumed pseudo-Riemannian manifold $(M^4, {}^4g)$, we can use its metric ${}^4g_{\mu\nu}$ to define the "orthonormal frame bundle" of M^4, $F(M^4) = M^4 \times SO(3,1)$, with structure group SO(3,1), of the orthonormal frames (or noncoordinate basis or orthonormal tetrads) ${}^4E_{(\alpha)} = {}^4E^\mu_{(\alpha)}\partial_\mu$ of TM^4. The orthonormal tetrads and their duals, the orthonormal cotetrads ${}^4E^{(\alpha)}_\mu$ [${}^4\theta^{(\alpha)} = {}^4E^{(\alpha)}_\mu dx^\mu$ are the orthonormal coframes], satisfy the duality and orthonormality conditions

$$\begin{aligned}
&{}^4E^{(\alpha)}_\mu\, {}^4E^\mu_{(\beta)} = \delta^{(\alpha)}_{(\beta)}, \qquad {}^4E^{(\alpha)}_\mu\, {}^4E^\nu_{(\alpha)} = \delta^\nu_\mu,\\
&{}^4E^\mu_{(\alpha)}\, {}^4g_{\mu\nu}\, {}^4E^\nu_{(\beta)} = {}^4\eta_{(\alpha)(\beta)}, \qquad {}^4E^{(\alpha)}_\mu\, {}^4g^{\mu\nu}\, {}^4E^{(\beta)}_\nu = {}^4\eta^{(\alpha)(\beta)}.
\end{aligned} \tag{11}$$

Under a rotation $\Lambda \in SO(3,1)$ [$\Lambda\, {}^4\eta\, \Lambda^T = {}^4\eta$] we have ${}^4E^\mu_{(\alpha)} \mapsto {}^4E^\mu_{(\beta)} (\Lambda^{-1})^{(\beta)}{}_{(\alpha)}$, ${}^4E^{(\alpha)}_\mu \mapsto \Lambda^{(\alpha)}{}_{(\beta)}\, {}^4E^{(\beta)}_\mu$. Therefore, while the indices α, β... transform under general coordinate transformations [the diffeomorphisms of $Diff\, M^4$], the indices $(\alpha), (\beta)$... transform under Lorentz rotations. The 4-metric can be expressed in terms of orthonormal cotetrads or local coframes in the noncoordinate basis

$$\begin{aligned}
&{}^4g_{\mu\nu} = {}^4E^{(\alpha)}_\mu\, {}^4\eta_{(\alpha)(\beta)}\, {}^4E^{(\beta)}_\nu, \qquad {}^4g^{\mu\nu} = {}^4E^\mu_{(\alpha)}\, {}^4\eta^{(\alpha)(\beta)}\, {}^4E^\nu_{(\beta)},\\
&{}^4g = {}^4g_{\mu\nu}\, dx^\mu \otimes dx^\nu = {}^4\eta_{(\alpha)(\beta)}\, \theta^{(\alpha)} \otimes \theta^{(\beta)}.
\end{aligned} \tag{12}$$

For each vector ${}^4V^\mu$ and covector ${}^4\omega_\mu$ we have the decompositions ${}^4V^\mu = {}^4V^{(\alpha)}\, {}^4E^\mu_{(\alpha)}$ [${}^4V^{(\alpha)} = {}^4E^{(\alpha)}_\mu\, {}^4V^\mu$], ${}^4\omega_\mu = {}^4E^{(\alpha)}_\mu\, {}^4\omega_{(\alpha)}$ [${}^4\omega_{(\alpha)} = {}^4E^\mu_{(\alpha)}\, {}^4\omega_\mu$].

In a noncoordinate (nonholonomic) basis we have

$$\begin{aligned}
&[{}^4E_{(\alpha)}, {}^4E_{(\beta)}] = c_{(\alpha)(\beta)}{}^{(\gamma)}\, {}^4E_{(\gamma)},\\
&c_{(\alpha)(\beta)}{}^{(\gamma)} = {}^4E^{(\gamma)}_\nu ({}^4E^\mu_{(\alpha)}\, \partial_\mu\, {}^4E^\nu_{(\beta)} - {}^4E^\mu_{(\beta)} \partial_\mu\, {}^4E^\nu_{(\alpha)}).
\end{aligned} \tag{13}$$

Physically, in a coordinate system (chart) x^μ of M^4, a tetrad may be considered as a collection of accelerated observers described by a congruence of timelike curves with 4-velocity ${}^4E^\mu_{(o)}$; in each point $p \in M^4$ consider a coordinate transformation to local inertial coordinates at p, i.e. $x^\mu \mapsto X^{(\mu)}_p(x)$: then we have, in p, ${}^4E^\mu_{(\alpha)}(p) = \frac{\partial x^\mu(X_p(p))}{\partial X^{(\alpha)}_p}$ and ${}^4E^{(\alpha)}_\mu(p) = \frac{\partial X^{(\alpha)}_p(p))}{\partial x^\mu}$ and locally we have a freely falling observer.

All the connection one-forms ω on the orthonormal frame bundle $F(M^4) = M^4 \times SO(3,1)$ have a torsion 2-form [it is $\mathcal{T} = \mathcal{D}^{(\omega)}\theta$, where θ is the canonical or soldering one-form (the coframes or cotetrads) and $\mathcal{D}^{(\omega)}$ is the $F(M^4)$ exterior covariant derivative], except the Levi-Civita connection ω_Γ. Therefore, since in general relativity we consider only Levi-Civita connections associated with pseudo-Riemannian 4-manifolds $(M^4, {}^4g)$, in $F(M^4)$ we consider only ω_Γ-horizontal subspaces H_Γ [$TF(M^4) = V_\Gamma + H_\Gamma$ as a direct sum, with V_Γ the vertical subspace isomorphic to the Lie algebra o(3,1) of SO(3,1)]. Given a global cross section $\sigma : M^4 \rightarrow F(M^4) = M^4 \times SO(3,1)$, the associated gauge potentials on M^4, ${}^4\omega = \sigma^*\omega$, are the connection coefficients ${}^4\omega^{(T)} = \sigma^*\omega$ in the noncoordinate basis ${}^4E_{(\alpha)}$ [the second line defines them through the covariant derivative in the noncoordinate basis]

$$
\begin{aligned}
&{}^4\omega^{(T)(\gamma)}_{(\alpha)(\beta)} = {}^4E^{(\gamma)}_{\nu}{}^4E^{\mu}_{(\alpha)}(\partial_\mu\, {}^4E^{\nu}_{(\beta)} + {}^4E^{\lambda}_{(\beta)}\, {}^4\Gamma^{(T)\nu}_{\mu\lambda}) = {}^4E^{(\gamma)}_{\nu}{}^4E^{\mu}_{(\alpha)}\, {}^4\nabla_\mu\, {}^4E^{\nu}_{(\beta)},\\
&{}^4\tilde{\nabla}_{{}^4E_{(\alpha)}}\, {}^4E_{(\beta)} = {}^4\nabla_{{}^4E_{(\alpha)}}\, {}^4E_{(\beta)} - {}^4\omega^{(T)(\gamma)}_{(\alpha)(\beta)}\, {}^4E_{(\gamma)} = 0.
\end{aligned}
\tag{14}
$$

The components of the Riemann tensors in the noncoordinate bases are ${}^4R^{(\alpha)}{}_{(\beta)(\gamma)(\delta)} = {}^4E_{(\gamma)}({}^4\omega^{(T)(\alpha)}_{(\delta)(\beta)}) - {}^4E_{(\delta)}({}^4\omega^{(T)(\alpha)}_{(\gamma)(\beta)}) + {}^4\omega^{(T)(\epsilon)}_{(\delta)(\beta)}\, {}^4\omega^{(T)(\alpha)}_{(\gamma)(\epsilon)} - {}^4\omega^{(T)(\epsilon)}_{(\gamma)(\beta)}\, {}^4\omega^{(T)(\alpha)}_{(\delta)(\epsilon)} - c_{(\gamma)(\delta)}{}^{(\epsilon)}\, {}^4\omega^{(T)(\alpha)}_{(\epsilon)(\beta)}$. The connection (gauge potential) one-form ${}^4\omega^{(T)(\alpha)}{}_{(\beta)} = {}^4\omega^{(T)(\alpha)}_{(\gamma)(\beta)}\, {}^4\theta^{(\gamma)}$ [it is called improperly "spin connection", while its components are called Ricci rotation coefficients] and the curvature (field strength) 2-form ${}^4\Omega^{(T)(\alpha)}{}_{(\beta)} = \frac{1}{2}{}^4\Omega^{(T)(\alpha)}{}_{(\beta)(\gamma)(\delta)}\, {}^4\theta^{(\gamma)} \wedge {}^4\theta^{(\delta)}$ satisfy the Cartan's structure equations

$$
\begin{aligned}
&d^4\theta^{(\alpha)} + {}^4\omega^{(T)(\alpha)}{}_{(\beta)} \wedge {}^4\theta^{(\beta)} = {}^4T^{(\alpha)},\\
&d^4\omega^{(T)(\alpha)}{}_{(\beta)} + {}^4\omega^{(T)(\alpha)}{}_{(\gamma)} \wedge {}^4\omega^{(T)(\gamma)}{}_{(\beta)} = {}^4\Omega^{(T)(\alpha)}{}_{(\beta)},
\end{aligned}
\tag{15}
$$

whose exterior derivatives $0 = d^4T^{(\alpha)} + {}^4\omega^{(T)(\alpha)}{}_{(\beta)} \wedge {}^4T^{(\beta)} = {}^4\Omega^{(T)(\alpha)}{}_{(\beta)} \wedge {}^4\theta^{(\beta)}$, $d^4\Omega^{(T)(\alpha)}{}_{(\beta)} + {}^4\omega^{(T)(\alpha)}{}_{(\gamma)} \wedge {}^4\Omega^{(T)(\gamma)}{}_{(\beta)} - {}^4\Omega^{(T)(\alpha)}{}_{(\gamma)} \wedge {}^4\omega^{(T)(\gamma)}{}_{(\beta)} \equiv 0$ are the two Bianchi identities.

With the Levi-Civita connection [which, as said, has zero torsion 2-form ${}^4T^{(\alpha)} = \frac{1}{2}T^{(\alpha)}{}_{(\beta)(\gamma)}\, {}^4\theta^{(\beta)} \wedge {}^4\theta^{(\gamma)} = 0$, namely ${}^4T^{(\alpha)}{}_{(\beta)(\gamma)} = {}^4\omega^{(T)(\alpha)}_{(\beta)(\gamma)} - {}^4\omega^{(T)(\alpha)}_{(\gamma)(\beta)} - c_{(\beta)(\gamma)}{}^{(\alpha)} = 0$], in a noncoordinate basis the spin connection takes the form

$$
\begin{aligned}
{}^4\omega^{(\alpha)}{}_{(\beta)} &= {}^4\omega^{(\alpha)}_{(\gamma)(\beta)}\, {}^4\theta^{(\gamma)} = {}^4\omega^{(\alpha)}_{\mu(\beta)} dx^\mu,\\
{}^4\omega_{(\alpha)(\gamma)(\beta)} &= {}^4\eta_{(\alpha)(\delta)}\, {}^4E^{(\delta)}_{\nu}\, {}^4E^{\mu}_{(\gamma)}\, {}^4\nabla_\mu\, {}^4E^{\nu}_{(\beta)} = {}^4\eta_{(\alpha)(\delta)}\, {}^4\omega^{(\delta)}_{(\gamma)(\beta)},\\
{}^4\omega^{(\alpha)}_{\mu(\beta)} &= {}^4\omega^{(\alpha)}_{(\gamma)(\beta)}\, {}^4E^{(\gamma)}_{\mu} = {}^4E^{(\alpha)}_{\nu}\, {}^4\nabla_\mu\, {}^4E^{\nu}_{(\beta)} = {}^4E^{(\alpha)}_{\nu}[\partial_\mu\, {}^4E^{\nu}_{(\beta)} + {}^4\Gamma^{\nu}_{\mu\rho}\, {}^4E^{\rho}_{(\beta)}],\\
\Rightarrow {}^4\Gamma^{\mu}_{\rho\sigma} &= \frac{1}{2}[{}^4E^{(\beta)}_{\sigma}({}^4E^{\mu}_{(\alpha)}\, {}^4E^{(\gamma)}_{\rho}\, {}^4\omega^{(\alpha)}_{(\gamma)(\beta)} - \partial_\rho\, {}^4E^{\mu}_{(\beta)}) +\\
&+ {}^4E^{(\beta)}_{\rho}({}^4E^{\mu}_{(\alpha)}\, {}^4E^{(\gamma)}_{\sigma}\, {}^4\omega^{(\alpha)}_{(\gamma)(\beta)} - \partial_\sigma\, {}^4E^{\mu}_{(\beta)})],
\end{aligned}
\tag{16}
$$

and the metric compatibility ${}^4\nabla_\rho\, {}^4g_{\mu\nu} = 0$ becomes the following condition

$$
{}^4\omega_{(\alpha)(\beta)} = {}^4\eta_{(\alpha)(\delta)}\, {}^4\omega^{(\delta)}{}_{(\beta)} = {}^4\eta_{(\alpha)(\delta)}{}^4\omega^{(\delta)}_{(\gamma)(\beta)}\, {}^4\theta^{(\gamma)} = {}^4\omega_{(\alpha)(\gamma)(\beta)}\, {}^4\theta^{(\gamma)} = -{}^4\omega_{(\beta)(\alpha)}
\tag{17}
$$

or ${}^4\omega_{(\alpha)(\gamma)(\beta)} = -{}^4\omega_{(\beta)(\gamma)(\alpha)}$ [${}^4\omega_{(\alpha)(\gamma)(\beta)}$ are called Ricci rotation coefficients, only 24 of which are independent]

Given a vector ${}^4V^\mu = {}^4V^{(\alpha)}\, {}^4E^{\mu}_{(\alpha)}$ and a covector ${}^4\omega_\mu = {}^4\omega_{(\alpha)}\, {}^4E^{(\alpha)}_{\mu}$, we define the covariant derivative of the components ${}^4V^{(\alpha)}$ and ${}^4\omega_{(\alpha)}$ as ${}^4\nabla_\nu\, {}^4V^\mu = {}^4V^\mu{}_{;\nu} \equiv [{}^4\nabla_\nu\, {}^4V^{(\alpha)}]\, {}^4E^{\mu}_{(\alpha)} = {}^4V^{(\alpha)}{}_{;\nu}\, {}^4E^{\mu}_{(\alpha)}$ and ${}^4\nabla_\nu\, {}^4\omega_\mu = {}^4\omega_{\mu;\nu} \equiv [{}^4\nabla_\nu\, {}^4\omega_{(\alpha)}]\, {}^4E^{(\alpha)}_{\mu} = {}^4\omega_{(\alpha);\nu}\, {}^4E^{(\alpha)}_{\mu}$, so that

$$
\begin{aligned}
{}^4V^{\mu}_{;\nu} &= \partial_\nu\, {}^4V^{(\alpha)}\, {}^4E^{\mu}_{(\alpha)} + {}^4V^{(\alpha)}\, {}^4E^{\mu}_{(\alpha);\nu},\\
&\Rightarrow {}^4V^{(\alpha)}{}_{;\nu} = \partial_\nu\, {}^4V^{(\alpha)} + {}^4\omega^{(\alpha)}_{\nu(\beta)}\, {}^4V^{(\beta)},\\
{}^4\omega_{\mu;\nu} &= \partial_\nu\, {}^4\omega_{(\alpha)}\, {}^4E^{(\alpha)}_{\mu} + {}^4\omega_{(\alpha)}\, {}^4E^{(\alpha)}_{\mu;\nu},\\
&\Rightarrow {}^4\omega_{(\alpha);\nu} = \partial_\nu\, {}^4\omega_{(\alpha)} - {}^4\omega_{(\beta)}\, {}^4\omega^{(\beta)}_{\nu(\alpha)}.
\end{aligned}
\tag{18}
$$

Therefore, for the "internal tensors" ${}^4T^{(\alpha)\dots}{}_{(\beta)\dots}$, the spin connection ${}^4\omega^{(\alpha)}_{\mu(\beta)}$ is a gauge potential associated with a gauge group SO(3,1). For internal vectors ${}^4V^{(\alpha)}$ at $p \in M^4$ the cotetrads ${}^4E^{(\alpha)}_\mu$ realize a soldering of this internal vector space at p with the tangent space T_pM^4: ${}^4V^{(\alpha)} = {}^4E^{(\alpha)}_\mu\,{}^4V^\mu$. For tensors with mixed world and internal indices, like tetrads and cotetrads, we could define a generalized covariant derivative acting on both types of indices ${}^4\tilde{\nabla}_\nu\,{}^4E^\mu_{(\alpha)} = \partial_\nu\,{}^4E^\mu_{(\alpha)} + {}^4\Gamma^\mu_{\nu\rho}\,{}^4E^\rho_{(\alpha)} - {}^4E^\mu_{(\beta)}\,{}^4\omega^{(\beta)}_{\nu(\alpha)}$: then ${}^4\nabla_\nu\,{}^4V^\mu = {}^4\nabla_\nu\,{}^4V^{(\alpha)}\,{}^4E^\mu_{(\alpha)} + {}^4V^{(\alpha)}\,{}^4\tilde{\nabla}_\nu\,{}^4E^\mu_{(\alpha)} \equiv {}^4\nabla_\nu\,{}^4V^{(\alpha)}\,{}^4E^\mu_{(\alpha)}$ implies ${}^4\tilde{\nabla}_\nu\,{}^4E^\mu_{(\alpha)} = 0$ [or ${}^4\nabla_\nu\,{}^3E^\mu_{(\alpha)} = {}^4E^\mu_{(\beta)}\,{}^4\omega^{(\beta)}_{\nu(\alpha)}$] which is nothing else that the definition (16) of the spin connection ${}^4\omega^{(\alpha)}_{\mu(\beta)}$.

We have

$$
\begin{aligned}
[{}^4E_{(\alpha)}, {}^4E_{(\beta)}] &= c_{(\alpha)(\beta)}{}^{(\gamma)}\,{}^4E_{(\gamma)} = {}^4\nabla_{{}^4E_{(\alpha)}}\,{}^4E_{(\beta)} - {}^4\nabla_{{}^4E_{(\beta)}}\,{}^4E_{(\alpha)} = \\
&= ({}^4\omega^{(\gamma)}_{(\alpha)(\beta)} - {}^4\omega^{(\gamma)}_{(\beta)(\alpha)})\,{}^4E_{(\gamma)},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
{}^4\Omega^{(\alpha)}{}_{(\beta)(\gamma)(\delta)} &= {}^4E_{(\gamma)}({}^4\omega^{(\alpha)}_{(\delta)(\beta)}) - {}^4E_{(\delta)}({}^4\omega^{(\alpha)}_{(\gamma)(\beta)}) + \\
&+ {}^4\omega^{(\epsilon)}_{(\delta)(\beta)}\,{}^4\omega^{(\alpha)}_{(\gamma)(\epsilon)} - {}^4\omega^{(\epsilon)}_{(\gamma)(\beta)}\,{}^4\omega^{(\alpha)}_{(\delta)(\epsilon)} - ({}^4\omega^{(\epsilon)}_{(\gamma)(\delta)} - {}^4\omega^{(\epsilon)}_{(\delta)(\gamma)})\,{}^4\omega^{(\alpha)}_{(\epsilon)(\beta)} = \\
&= {}^4E^{(\alpha)}_\mu\,{}^4R^\mu{}_{\rho\nu\sigma}\,{}^4E^\rho_{(\beta)}\,{}^4E^\nu_{(\gamma)}\,{}^4E^\sigma_{(\delta)},
\end{aligned}
$$

$$
\begin{aligned}
{}^4\Omega_{\mu\nu}{}^{(\alpha)}{}_{(\beta)} &= {}^4E^{(\gamma)}_\mu\,{}^4E^{(\delta)}_\nu\,{}^4\Omega^{(\alpha)}{}_{(\beta)(\gamma)(\delta)} = {}^4R^\rho{}_{\sigma\mu\nu}\,{}^4E^{(\alpha)}_\rho\,{}^4E^\sigma_{(\beta)} = \\
&= \partial_\mu\,{}^4\omega^{(\alpha)}_{\nu(\beta)} - \partial_\nu\,{}^4\omega^{(\alpha)}_{\mu(\beta)} + {}^4\omega^{(\alpha)}_{\mu(\gamma)}\,{}^4\omega^{(\gamma)}_{\nu(\beta)} - {}^4\omega^{(\alpha)}_{\nu(\gamma)}\,{}^4\omega^{(\gamma)}_{\mu(\beta)},\\
{}^4\Omega_{\mu\nu(\alpha)(\beta)} &= {}^4\eta_{(\alpha)(\gamma)}\,{}^4\Omega_{\mu\nu}{}^{(\gamma)}{}_{(\beta)} = -{}^4\Omega_{\nu\mu(\alpha)(\beta)} = -{}^4\Omega_{\mu\nu(\beta)(\alpha)},
\end{aligned}
$$

$$
\begin{aligned}
{}^4R^\alpha{}_{\beta\mu\nu} &= {}^4E^\alpha_{(\gamma)}\,{}^4E^{(\delta)}_\beta\,{}^4\Omega_{\mu\nu}{}^{(\gamma)}{}_{(\delta)},\\
{}^4R_{\mu\nu} &= {}^4E^\alpha_{(\gamma)}\,{}^4E^{(\delta)}_\nu\,{}^4\Omega_{\alpha\mu}{}^{(\gamma)}{}_{(\delta)},\\
{}^4R &= {}^4E^\mu_{(\gamma)}\,{}^4E^{(\delta)}_\rho\,{}^4g^{\rho\nu}\,{}^4\Omega_{\mu\nu}{}^{(\gamma)}{}_{(\delta)},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&d^4\theta^{(\alpha)} + {}^4\omega^{(\alpha)}{}_{(\beta)} \wedge {}^4\theta^{(\beta)} = 0,\\
&d^4\omega^{(\alpha)}{}_{(\beta)} + {}^4\omega^{(\alpha)}{}_{(\gamma)} \wedge {}^4\omega^{(\gamma)}{}_{(\beta)} = {}^4\Omega^{(\alpha)}{}_{(\beta)},
\end{aligned}
\tag{21}
$$

with the Bianchi identities ${}^4\Omega^{(\alpha)}{}_{(\beta)} \wedge {}^4\theta^{(\beta)} \equiv 0$, $d^4\Omega^{(\alpha)}{}_{(\beta)} + {}^4\omega^{(\alpha)}{}_{(\gamma)} \wedge {}^4\Omega^{(\gamma)}{}_{(\beta)} - {}^4\Omega^{(\alpha)}{}_{(\gamma)} \wedge {}^4\omega^{(\gamma)}{}_{(\beta)} \equiv 0$.

Let us remark that Eqs.(14) and (16) imply ${}^4\Gamma^\rho_{\mu\nu} = {}^4\triangle^\rho_{\mu\nu} + {}^4\omega^\rho_{\mu\nu}$ with ${}^4\omega^\rho_{\mu\nu} = {}^4E^\rho_{(\alpha)}\,{}^4E^{(\beta)}_\nu\,{}^4\omega^{(\alpha)}_{\mu(\beta)}$ and ${}^4\triangle^\rho_{\mu\nu} = {}^4E^\rho_{(\alpha)}\,\partial_\mu\,{}^4E^{(\alpha)}_\nu$; the Levi-Civita connection (i.e. the Christoffel symbols) turn out to be decomposed in a flat connection ${}^4\triangle^\rho_{\mu\nu}$ (it produces zero Riemann tensor as was already known to Einstein [75]) and in a tensor, like in the Yang-Mills case [5].

Let us finish this Section with a review of some action principles used for general relativity. In metric gravity, one uses the generally covariant Hilbert action depending on the 4-metric and its first and second derivatives [G is Newton gravitational constant; $U \subset M^4$ is a subset of spacetime; we use units with $x^o = ct$]

$$
S_H = \frac{c^3}{16\pi G}\int_U d^4x\,\sqrt{{}^4g}\,{}^4R = \int_U d^4x\,\mathcal{L}_H.
\tag{22}
$$

The variation of S_H is $[d^3\Sigma_\gamma = d^3\Sigma l_\gamma]$

$$\delta S_H = \delta S_E + \Sigma_H = -\frac{c^3}{16\pi G}\int_U d^4x\,\sqrt{{}^4g}\,{}^4G^{\mu\nu}\delta^4 g_{\mu\nu} + \Sigma_H,$$

$$\Sigma_H = \frac{c^3}{16\pi G}\int_{\partial U} d^3\Sigma_\gamma \sqrt{{}^4g}\,({}^4g^{\mu\nu}\delta^\gamma_\delta - {}^4g^{\mu\gamma}\delta^\nu_\delta)\delta\,{}^4\Gamma^\delta_{\mu\nu} =$$
$$= \frac{c^3}{8\pi G}\int_{\partial U} d^3\Sigma\,\sqrt{{}^3\gamma}\;\delta\,{}^3K,$$

$$\delta^4\Gamma^\delta_{\mu\nu} = \frac{1}{2}{}^4g^{\delta\beta}[{}^4\nabla_\mu\delta\,{}^4g_{\beta\nu} + {}^4\nabla_\nu\delta\,{}^4g_{\beta\mu} - {}^4\nabla_\beta\delta\,{}^4g_{\mu\nu}]. \tag{23}$$

where ${}^3\gamma_{\mu\nu}$ is the metric induced on ∂U and l_μ is the outer unit covariant normal to ∂U. The trace of the extrinsic curvature ${}^3K_{\mu\nu}$ of ∂U is ${}^3K = -l^\mu{}_{;\mu}$. The surface term Σ_H takes care of the second derivatives of the 4-metric and to get Einstein equations ${}^4G_{\mu\nu} = {}^4R_{\mu\nu} - \frac{1}{2}{}^4g_{\mu\nu}\,{}^4R \circeq 0$ one must take constant certain normal derivatives of the 4-metric on the boundary of U $[\mathcal{L}_l\,({}^4g_{\mu\nu} - l_\mu l_\nu) = 0]$ to have $\delta S_H = 0$ [76].

The term δS_E in Eq.(23) means the variation of the action S_E, which is the (not generally covariant) Einstein action depending only on the 4-metric and its first derivatives $[\delta S_E = 0$ gives ${}^4G_{\mu\nu} \circeq 0$ if ${}^4g_{\mu\nu}$ is held fixed on $\partial U]$

$$S_E = \int_U d^4x\,\mathcal{L}_E = \frac{c^3}{16\pi G}\int_U d^4x\,\sqrt{{}^4g}\,{}^4g^{\mu\nu}({}^4\Gamma^\rho_{\nu\lambda}\,{}^4\Gamma^\lambda_{\rho\mu} - {}^4\Gamma^\lambda_{\lambda\rho}\,{}^4\Gamma^\rho_{\mu\nu}) =$$
$$= S_H - \frac{c^3}{16\pi G}\int_U d^4x\,\partial_\lambda[\sqrt{{}^4g}({}^4g^{\mu\nu}\,{}^4\Gamma^\lambda_{\mu\nu} - {}^4g^{\lambda\mu}\,{}^4\Gamma^\rho_{\rho\mu})],$$

$$\delta S_E = \frac{c^3}{16\pi G}\int_U d^4x\,(\frac{\partial\mathcal{L}_E}{\partial\,{}^4g^{\mu\nu}} - \partial_\rho\,\frac{\partial\mathcal{L}_E}{\partial\partial_\rho\,{}^4g^{\mu\nu}})\,\delta\,{}^4g^{\mu\nu} = -\frac{c^3}{16\pi G}\int_U d^4x\,\sqrt{{}^4g}\,{}^4G_{\mu\nu}\delta\,{}^4g^{\mu\nu}. \tag{24}$$

We shall not consider the first-order Palatini action; see for instance Ref. [77], where there is also a review of the variational principles of the connection-dependent formulations of general relativity.

In Ref. [76] (see also Ref. [73]), it is shown that the DeWitt-ADM action [78,32] for a 3+1 decomposition of M^4 can be obtained from S_H in the following way $[\sqrt{{}^4g}{}^4R = -\epsilon\sqrt{{}^4g}({}^3R + {}^3K_{\mu\nu}{}^3K^{\mu\nu} - ({}^3K)^2) - 2\epsilon\,\partial_\lambda(\sqrt{{}^4g}({}^3Kl^\lambda + a^\lambda))$, with a^λ the 4-acceleration $(l^\mu a_\mu = 0)$; the 4-volume U is $[\tau_f,\tau_i]\times S]$

$$S_H = S_{ADM} + \Sigma_{ADM},$$
$$S_{ADM} = -\epsilon\frac{c^3}{16\pi G}\int_U d^4x\,\sqrt{{}^4g}[{}^3R + {}^3K_{\mu\nu}\,{}^3K^{\mu\nu} - ({}^3K)^2],$$
$$\Sigma_{ADM} = -\epsilon\frac{c^3}{8\pi G}\int d^4x\partial_\alpha[\sqrt{{}^4g}({}^3Kl^\alpha + l^\beta l^\alpha{}_{;\beta})] =$$
$$= -\epsilon\frac{c^3}{8\pi G}\Big[\int_S d^3\sigma\,[\sqrt{\gamma}\;{}^3K](\tau,\vec{\sigma})|^{\tau_f}_{\tau_i} +$$
$$+\int_{\tau_i}^{\tau_f} d\tau\int_{\partial S} d^2\Sigma^r[{}^3\nabla_r(\sqrt{\gamma}N) - {}^3KN_r](\tau,\vec{\sigma})\Big],$$

$$
\begin{aligned}
&\delta S_{ADM} = -\epsilon \frac{c^3}{16\pi G} \int d\tau d^3\sigma \sqrt{\gamma} \Big[2\, {}^4\bar{G}_{ll} \delta N + {}^4\bar{G}_l{}^r \delta N_r - {}^4\bar{G}^{rs} \delta\, {}^3g_{rs} \Big](\tau,\vec{\sigma}) +\\
&+ \delta S_{ADM}|_{{}^4G_{\mu\nu}=0} - \epsilon \int_{\tau_i}^{\tau_f} d\tau \int_{\partial U} d^3\Sigma_r [N_{|s} \delta\, {}^3g^{rs} - N \delta\, {}^3g^{rs}{}_{|s}](\tau,\vec{\sigma}),\\
&\delta S_{ADM}|_{{}^4G_{\mu\nu}=0} = -\epsilon \frac{c^3}{16\pi G} \int_{\partial U} d^3\sigma\, {}^3\tilde{\Pi}^{\mu\nu} \delta^3\gamma_{\mu\nu},\\
&{}^3\tilde{\Pi}^{\mu\nu} = \sqrt{\gamma} ({}^3K^{\mu\nu} - {}^3g^{\mu\nu}\, {}^3K) = \frac{16\pi G}{c^3} \epsilon \hat{b}^\mu_r \hat{b}^\nu_s\, {}^3\tilde{\Pi}^{rs},
\end{aligned} \tag{25}
$$

so that $\delta S_{ADM} = 0$ gives ${}^4G_{\mu\nu} \overset{\circ}{=} 0$ if one holds fixed the intrinsic 3-metric ${}^3\gamma_{\mu\nu}$ on the boundary [${}^3\tilde{\Pi}^{\mu\nu}$ is the ADM momentum with world indices, whose form in a 3+1 splitting is given in Section V]. This action is not generally covariant, but it is quasi-invariant under the 8 types of gauge transformations generated by the ADM first class constraints, as it will be shown in the third paper of the series. As shown in Refs. [80,33,76,81] in this way one obtains a well defined gravitational energy. However, in so doing one still neglects some boundary terms. Following Ref. [81], let us assume that, given a subset $U \subset M^4$ of spacetime, ∂U consists of two slices, Σ_{τ_i} (the initial one) and Σ_{τ_f} (the final one) with outer normals $-l^\mu(\tau_i,\vec{\sigma})$ and $l^\mu(\tau_f,\vec{\sigma})$ respectively, and of a surface S_∞ near space infinity with outer unit (spacelike) normal $n^\mu(\tau,\vec{\sigma})$ tangent to the slices [so that the normal $l^\mu(\tau,\vec{\sigma})$ to every slice is asymptotically tangent to S_∞]. The 3-surface S_∞ is foliated by a family of 2-surfaces $S^2_{\tau,\infty}$ coming from its intersection with the slices Σ_τ [therefore, asymptotically $l^\mu(\tau,\vec{\sigma})$ is normal to the corresponding $S^2_{\tau,\infty}$]. The vector $b^\mu_\tau = z^\mu_\tau = N l^\mu + N^r b^\mu_r$ is not in general tangent to S_∞. It is assumed that there are no inner boundaries (see Ref. [81] for their treatment), so that the slices Σ_τ do not intersect and are complete. This does not rule out the existence of horizons, but it implies that, if horizons form, one continues to evolve the spacetime inside the horizon as well as outside. Then, in Ref. [81] it is shown that one gets [${}^2K$ the trace of the 2-dimensional extrinsic curvature of the 2-surface $S^2_{\tau,\infty} = S_\infty \cap \Sigma_\tau$; to get this result one assumes that the lapse function $N(\tau,\vec{\sigma})$ on Σ_τ tends asymptotically to a function $N_{(as)}(\tau)$ and that the term on ∂S vanishes due to the boundary conditions]

$$
\begin{aligned}
\Sigma_{ADM} &= -\epsilon \frac{c^3}{8\pi G} [\int_{\Sigma_{\tau_f}} d^3\Sigma - \int_{\Sigma_{\tau_i}} d^3\Sigma]\, N \sqrt{\gamma}\, {}^3K =\\
&= -\epsilon \frac{c^3}{8\pi G} \int_{\tau_i}^{\tau_f} d\tau\, N_{(as)}(\tau) \int_{S^2_{\tau,\infty}} d^2\Sigma \sqrt{\gamma}\, {}^2K.
\end{aligned} \tag{26}
$$

Instead, in tetrad gravity [58–63,65–69], in which ${}^4g_{\mu\nu}$ is no more the independent variable, the new independent 16 variables are a set of cotetrads ${}^4E^{(\alpha)}_\mu$ so that ${}^4g_{\mu\nu} = {}^4E^{(\alpha)}_\mu\, {}^4\eta_{(\alpha)(\beta)}\, {}^4E^{(\beta)}_\nu$. Tetrad gravity has not only the invariance under $Diff\, M^4$ but also under local Lorentz transformations on TM^4 [acting on the flat indices (α)]. An action principle with these local invariances is obtained by replacing the 4-metric in the Hilbert action S_H with its expression in terms of the cotetrads. The action acquires the form

$$
S_{HT} = \frac{c^3}{16\pi G} \int_U d^4x\, {}^4\tilde{E}\, {}^4E^\mu_{(\alpha)}\, {}^4E^\nu_{(\beta)}\, {}^4\Omega_{\mu\nu}{}^{(\alpha)(\beta)}, \tag{27}
$$

where ${}^4\tilde{E} = det\, ({}^4E^{(\alpha)}_\mu) = \sqrt{{}^4g}$ and ${}^4\Omega_{\mu\nu}{}^{(\alpha)(\beta)}$ is the spin 4-field strength . One has

$$\delta S_{HT} = \frac{c^3}{16\pi G} \int_U d^4x\, {}^4\tilde{E}\, {}^4G_{\mu\nu}\, {}^4E^\mu_{(\alpha)}\, {}^4\eta^{(\alpha)(\beta)}\, \delta\, {}^4E^\nu_{(\beta)} +$$
$$+ \frac{c^3}{8\pi G} \int_U d^4x\, \partial_\mu [{}^4\tilde{E}\, ({}^4E^{(\rho)}_\nu \delta({}^4g^{\mu\lambda}\, {}^4\nabla_\lambda {}^4E^\nu_{(\rho)}) - {}^4\eta^{(\rho)(\sigma)}\, {}^4E^\nu_{(\rho)} \delta({}^4\nabla_\nu\, {}^4E^\mu_{(\sigma)}))]. \qquad (28)$$

Again $\delta S_{HT} = 0$ produces Einstein equations if complicated derivatives of the tetrads vanish at the boundary. In Ref. [63], by using ${}^4\tilde{E} {}^4E^\mu_{(\alpha)}\, {}^4E^\nu_{(\beta)}\, {}^4\Omega_{\mu\nu}{}^{(\alpha)(\beta)} = 2\, {}^4\tilde{E}\, {}^4E^\mu_{(\alpha)}\, {}^4E^\nu_{(\beta)} [{}^4\omega_\mu\, {}^4\omega_\nu - {}^4\omega_\nu\, {}^4\omega_\mu]^{(\alpha)(\beta)} + 2\, \partial_\mu ({}^4\tilde{E}\, {}^4E^\mu_{(\alpha)}\, {}^4E^\nu_{(\beta)}\, {}^4\omega_\nu{}^{(\alpha)(\beta)})$, the analogue of S_E, i.e. the (not locally Lorentz invariant, therefore not expressible only in terms of the 4-metric) Charap action, is defined as

$$S_C = -\frac{c^3}{8\pi G} \int_U d^4x\, {}^4\tilde{E}\, {}^4E^\mu_{(\alpha)}\, {}^4E^\nu_{(\beta)} ({}^4\omega_\mu\, {}^4\omega_\nu - {}^4\omega_\nu\, {}^4\omega_\mu)^{(\alpha)(\beta)}. \qquad (29)$$

Its variation δS_C vanishes if $\delta\, {}^4E^\mu_{(\alpha)}$ vanish at the boundary and the Einstein equations hold. However its Hamiltonian formulation gives too complicated first class constraints to be solved.

In Einstein metric gravity the gravitational field, described by the 4-metric ${}^4g_{\mu\nu}$ depends on 2, and not 10, physical degrees of freedom in each point; this is not explicitly evident if one starts with the Hilbert action, which is invariant under $Diff\, M^4$, a group with only four generators. Instead in ADM canonical gravity (see Section V) there are in each point 20 canonical variables and 8 first class constraints, implying the determination of 8 canonical variables and the arbitrariness of the 8 conjugate ones. At the Lagrangian level, only 6 of the ten Einstein equations are independent, due to the contracted Bianchi identities, so that four components of the metric tensor ${}^4g_{\mu\nu}$ (the lapse and shift functions) are arbitrary not being determined by the equation of motion. Moreover, the four combinations ${}^4\bar{G}_{ll} \circeq 0$, ${}^4\bar{G}_{lr} \circeq 0$, of the Einstein equations do not depend on the second time derivatives or accelerations (they are restrictions on the Cauchy data and become the secondary first class constraints of the ADM canonical theory): the general theory [3] implies that four generalized velocities (and therefore other four components of the metric) inherit the arbitrariness of the lapse and shift functions. Only two combinations of the Einstein equations depend on the accelerations (second time derivatives) of the two (non tensorial) independent degrees of freedom of the gravitational field and are genuine equations of motion. Therefore, the ten components of every 4-metric ${}^4g_{\mu\nu}$, compatible with the Cauchy data, depend on 8 arbitrary functions not determined by the Einstein equations.

Tetrad gravity with action S_{HT}, in which the elementary natural Lagrangian object is the soldering or canonical one-form (or orthogonal coframe) $\theta^{(\alpha)} = {}^4E^{(\alpha)}_\mu dx^\mu$, is gauge invariant simultaneously under diffeomorphisms [$Diff\, M^4$] and Lorentz transformations [SO(3,1)]. Instead in phase space (see Section IV) only two of the 16 components of the cotetrad ${}^4E^{(\alpha)}_\mu(x)$ are physical degrees of freedom in each point, since the 32 canonical variables present in each point are restricted by 14 first class constraints, so that the 16 components of a cotetrad compatible with the Cauchy data depend on 14 arbitrary functions not determined by the equation of motion.

The gauge transformations of tetrad gravity with action S_{HT} are [$x^\mu \mapsto x'^\mu(x)$, $\Lambda(x) \in SO(3,1)$ for each $x^\mu \in M^4$]

$${}^4E^{(\alpha)}_\mu(x) \mapsto {}^4E'^{(\alpha)}_\mu(x'(x)) = \frac{\partial x^\nu}{\partial x'^\mu} \Lambda^{(\alpha)}{}_{(\beta)}(x)\, {}^4E^{(\beta)}_\nu(x),$$

$$
\begin{aligned}
{}^4g_{\mu\nu}(x) &\mapsto {}^4g^{'}_{\mu\nu}(x^{'}(x)) = \frac{\partial x^\alpha}{\partial x^{'\mu}}\frac{\partial x^\beta}{\partial x^{'\nu}}\,{}^4g_{\alpha\beta}(x),\\
{}^4\Gamma^\mu_{\alpha\beta}(x) &\mapsto {}^4\Gamma^{'\mu}_{\alpha\beta}(x^{'}(x)) = \frac{\partial x^{'\mu}}{\partial x^\nu}\frac{\partial x^\gamma}{\partial x^{'\alpha}}\frac{\partial x^\delta}{\partial x^{'\beta}}\,{}^4\Gamma^\nu_{\gamma\delta}(x) + \frac{\partial^2 x^\nu}{\partial x^{'\alpha}\partial x^{'\beta}}\frac{\partial x^{'\mu}}{\partial x^\nu},\\
{}^4R^\mu{}_{\alpha\nu\beta}(x) &\mapsto {}^4R^{'\mu}{}_{\alpha\nu\beta}(x^{'}(x)) = \frac{\partial x^{'\mu}}{\partial x^\rho}\frac{\partial x^\gamma}{\partial x^{'\alpha}}\frac{\partial x^\sigma}{\partial x^{'\nu}}\frac{\partial x^\delta}{\partial x^{'\beta}}\,{}^4R^\rho{}_{\gamma\sigma\delta}(x),\\
{}^4\omega^{(\alpha)}_{(\gamma)(\beta)}(x) &\mapsto {}^4\omega^{'(\alpha)}_{(\gamma)(\beta)}(x^{'}(x)) = \Lambda^{(\alpha)}{}_{(\mu)}(x)(\Lambda^{-1})^{(\rho)}{}_{(\gamma)}(x)(\Lambda^{-1})^{(\sigma)}{}_{(\beta)}(x)\,{}^4\omega^{(\mu)}_{(\rho)(\sigma)}(x) +\\
&+\ (\Lambda^{-1})^{(\rho)}{}_{(\gamma)}(x)\,{}^4E^\nu_{(\rho)}(x)\,\partial_\nu\,\Lambda^{(\alpha)}{}_{(\mu)}(x)\,(\Lambda^{-1})^{(\mu)}{}_{(\beta)}(x),\\
{}^4\Omega^{(\alpha)}{}_{(\beta)(\gamma)(\delta)}(x) &\mapsto {}^4\Omega^{'(\alpha)}{}_{(\beta)(\gamma)(\delta)}(x^{'}(x)) =\\
&=\ \Lambda^{(\alpha)}{}_{(\mu)}(x)\,(\Lambda^{-1})^{(\rho)}{}_{(\beta)}(x)\,(\Lambda^{-1})^{(\nu)}{}_{(\gamma)}(x)\,(\Lambda^{-1})^{(\sigma)}{}_{(\delta)}(x)\,{}^4\Omega^{(\mu)}{}_{(\rho)(\nu)(\sigma)}(x).
\end{aligned}
\tag{30}
$$

With the Lie derivative one can characterize the action of infinitesimal diffeomorphisms

$$
x^{'\mu}(x) = x^\mu + \xi^\mu(x) = x^\mu + \delta_o x^\mu, \qquad \Rightarrow x^\mu(x^{'}) \approx x^{'\mu} - \xi^\mu(x^{'}),
$$

$$
\begin{aligned}
\delta\,{}^4E^\mu_{(\alpha)}(x) &= {}^4E^{'\mu}_{(\alpha)}(x^{'}(x)) - {}^4E^\mu_{(\alpha)}(x) = \delta_o\,{}^4E^\mu_{(\alpha)}(x) + \xi^\nu(x)\partial_\nu\,{}^4E^\mu_{(\alpha)}(x) =\\
&= \frac{\partial x^{'\mu}}{\partial x^\nu}\,{}^4E^\nu_{(\alpha)}(x) - {}^4E^\mu_{(\alpha)}(x) = \partial_\nu\xi^\mu(x)\,{}^4E^\nu_{(\alpha)}(x),\\
\delta_o\,{}^4E^\mu_{(\alpha)}(x) &= {}^4E^{'\mu}_{(\alpha)}(x) - {}^4E^\mu_{(\alpha)}(x) = [\partial_\nu\xi^\mu(x) - \delta^\mu_\nu\xi^\rho(x)\partial_\rho]\,{}^4E^\nu_{(\alpha)}(x) = [\mathcal{L}_{-\xi^\rho\partial_\rho}\,{}^4E^\nu_{(\alpha)}(x)\partial_\nu]^\mu,
\end{aligned}
$$

$$
\begin{aligned}
\delta\,{}^4E^{(\alpha)}_\mu(x) &= {}^4E^{'(\alpha)}_\mu(x^{'}(x)) - {}^4E^{(\alpha)}_\mu(x) = \delta_o\,{}^4E^{(\alpha)}_\mu + \xi^\nu(x)\partial_\nu\,{}^4E^{(\alpha)}_\mu(x) =\\
&= \frac{\partial x^\nu}{\partial x^{'\mu}}\,{}^4E^{(\alpha)}_\nu(x) - {}^4E^{(\alpha)}_\mu(x) = -\partial_\mu\xi^\nu(x)\,{}^4E^{(\alpha)}_\nu(x),\\
\delta_o\,{}^4E^{(\alpha)}_\mu(x) &= {}^4E^{'(\alpha)}_\mu(x) - {}^4E^{(\alpha)}_\mu(x) = -[\partial_\mu\xi^\nu(x) + \delta^\nu_\mu\xi^\rho(x)\partial_\rho]\,{}^4E^{(\alpha)}_\nu(x) =\\
&= [\mathcal{L}_{-\xi^\rho\partial_\rho}\,{}^4E^{(\alpha)}_\nu(x)dx^\nu]_\mu,
\end{aligned}
$$

$$
\begin{aligned}
\delta\,{}^4g_{\mu\nu}(x) &= {}^4g^{'}_{\mu\nu}(x^{'}(x)) - {}^4g_{\mu\nu}(x) = \delta_o\,{}^4g_{\mu\nu}(x) + \xi^\rho(x)\partial_\rho\,{}^4g_{\mu\nu}(x) =\\
&= \frac{\partial x^\alpha}{\partial x^{'\mu}}\frac{\partial x^\beta}{\partial x^{'\nu}}\,{}^4g_{\alpha\beta}(x) - {}^4g_{\mu\nu}(x) = -[\delta^\alpha_\mu\partial_\nu\xi^\beta(x) + \delta^\beta_\nu\partial_\mu\xi^\alpha(x)]\,{}^4g_{\alpha\beta}(x),\\
\delta_o\,{}^4g_{\mu\nu}(x) &= {}^4g^{'}_{\mu\nu}(x) - {}^4g_{\mu\nu}(x) = -[\delta^\alpha_\mu\partial_\nu\xi^\beta(x) + \delta^\beta_\nu\partial_\mu\xi^\alpha(x) + \delta^\alpha_\mu\delta^\beta_\nu\xi^\rho(x)\partial_\rho]\,{}^4g_{\alpha\beta}(x) =\\
&= -[{}^4\nabla_\mu\xi_\nu(x) + {}^4\nabla_\nu\xi_\mu(x)] = [\mathcal{L}_{-\xi^\rho\partial_\rho}\,{}^4g_{\alpha\beta}dx^\alpha\otimes dx^\beta]_{\mu\nu}.
\end{aligned}
\tag{31}
$$

With the spin connection coefficients ${}^4\omega^{(\alpha)}_{\mu(\beta)} = {}^4\omega^{(\alpha)}_{(\gamma)(\beta)}\,{}^4E^{(\gamma)}_\mu$, and the field strengths ${}^4\Omega_{\mu\nu}{}^{(\alpha)}{}_{(\beta)}$, we get the transformation properties of the gauge potentials and field strengths of a SO(3,1) connection on the orthonormal frame bundle $F(M^4)$

$$
\begin{aligned}
{}^4\omega^{(\alpha)}_{\mu(\beta)}(x) &\mapsto {}^4\omega^{'(\alpha)}_{\mu(\beta)}(x^{'}(x)) = \frac{\partial x^\nu}{\partial x^{'\mu}}[\Lambda(x)\,{}^4\omega_\nu(x)\Lambda^{-1}(x) + \partial_\nu\,\Lambda(x)\Lambda^{-1}(x)]^{(\alpha)}{}_{(\beta)},\\
{}^4\Omega_{\mu\nu}{}^{(\alpha)}{}_{(\beta)}(x) &\mapsto {}^4\Omega^{'}_{\mu\nu}{}^{(\alpha)}{}_{(\beta)}(x^{'}(x)) = \frac{\partial x^\rho}{\partial x^{'\mu}}\frac{\partial x^\sigma}{\partial x^{'\nu}}\Lambda^{(\alpha)}{}_{(\gamma)}(x)\,{}^4\Omega_{\rho\sigma}{}^{(\gamma)}{}_{(\delta)}(x)\,(\Lambda^{-1})^{(\delta)}{}_{(\beta)}(x).
\end{aligned}
\tag{32}
$$

Instead in Refs. [66–69] it was implicitly used the metric ADM action $S_{ADM}[{}^4g_{\mu\nu}]$ with the metric expressed in terms of cotetrads in the Schwinger time gauge [60] as independent Lagrangian variables $S_{ADMT}[{}^4E^{(\alpha)}_{\mu}]$. This is the action we shall study in this paper after having expressed arbitrary cotetrads in terms of Σ_τ-adapted ones in the next Section.

Like S_{ADM} is not manifestly invariant under $Diff\, M^4$, also S_{ADMT} is not manifestly invariant under the transformations of Eqs.(30). However both theories are quasi-invariant under the gauge transformations generated by their first class constraints. This aspect of the theory, till now poorly explored due to the prevalence of the idea of general covariance, is the fundamental one in the presymplectic approach on which our discussion is based. A completely open point is the physical relevance of the special canonical Shanmugadhasan coordinate systems adapted to the constraints in generally covariant theories, since only in these coordinate systems there is a manifest (even if not tensorial) identification of which are the degrees of freedom, underlying general covariance, which are left undetermined by Einstein equations. The physical meaning of the so called gauge variables (conjugate to the Abelianized first class constraints) and of the resulting Dirac's observables is an open problem in theories with general covariance (it points at the existence of privileged structures natural from the presymplectic point of view, at least for noncompact spacetimes) on which we shall return in the next paper (no such problem exists with 'internal' gauge invariances like in Yang-Mills theory).

In what follows we shall use the notation $k = \frac{c^3}{16\pi G}$.

III. Σ_τ-ADAPTED TETRADS AND TRIADS.

On Σ_τ with local coordinate system $\{\sigma^r\}$ and Riemannian metric ${}^3g_{rs}$ of signature $(+++)$ we can introduce orthonormal frames (triads) ${}^3e_{(a)} = {}^3e^r_{(a)} \frac{\partial}{\partial \sigma^r}$, a=1,2,3, and coframes (cotriads) ${}^3\theta^{(a)} = {}^3e^{(a)}_r d\sigma^r$ satisfying

$$
\begin{aligned}
&{}^3e^r_{(a)}\, {}^3g_{rs}\, {}^3e^s_{(b)} = \delta_{(a)(b)}, \qquad {}^3e^{(a)}_r\, {}^3g^{rs}\, {}^3e^{(b)}_s = \delta^{(a)(b)},\\
&{}^3e^r_{(a)}\, \delta^{(a)(b)}\, {}^3e^s_{(b)} = {}^3g^{rs}, \qquad {}^3e^{(a)}_r\, \delta_{(a)(b)}\, {}^3e^{(b)}_s = {}^3g_{rs}.
\end{aligned} \tag{33}
$$

and consider the orthonormal frame bundle $F(\Sigma_\tau)$ over Σ_τ with structure group SO(3). See Ref. [82] for geometrical properties of triads.

The 3-dimensional spin connection 1-form ${}^3\omega^{(a)}_{r(b)} d\sigma^r$ is

$$
\begin{aligned}
{}^3\omega^{(a)}_{r(b)} &= {}^3\omega^{(a)}_{(c)(b)}\, {}^3e^{(c)}_r = {}^3e^{(a)}_s\, {}^3\nabla_r\, {}^3e^s_{(b)} =\\
&= {}^3e^{(a)}_s\, {}^3e^s_{(b)|r} = {}^3e^{(a)}_s [\partial_r\, {}^3e^s_{(b)} + {}^3\Gamma^s_{ru}\, {}^3e^u_{(b)}],
\end{aligned}
$$

$$
\begin{aligned}
{}^3\omega_{(a)(b)} &= \delta_{(a)(c)}\, {}^3\omega^{(c)}_{r(b)} d\sigma^r = -{}^3\omega_{(b)(a)}, \qquad {}^3\omega_{r(a)} = \frac{1}{2}\epsilon_{(a)(b)(c)}\, {}^3\omega_{r(b)(c)},\\
{}^3\omega_{r(a)(b)} &= \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(c)} = [\hat{R}^{(c)3}\omega_{r(c)}]_{(a)(b)} = [{}^3\omega_r]_{(a)(b)},
\end{aligned}
$$

$$
[{}^3e_{(a)}, {}^3e_{(b)}] = ({}^3\omega^{(c)}_{(a)(b)} - {}^3\omega^{(c)}_{(b)(a)})^3e_{(c)}, \tag{34}
$$

where $\epsilon_{(a)(b)(c)}$ is the standard Euclidean antisymmetric tensor and $(\hat{R}^{(c)})_{(a)(b)} = \epsilon_{(a)(b)(c)}$ is the adjoint representation of SO(3) generators.

Given vectors and covectors ${}^3V^r = {}^3V^{(a)}\, {}^3e^r_{(a)}$, ${}^3V_r = {}^3V_{(a)}\, {}^3e^{(a)}_r$, we have [remember that ${}^3\nabla_s\, {}^3e^r_{(a)} = {}^3e^r_{(b)}\, {}^3\omega^{(b)}_{s(a)}$]

$$
\begin{aligned}
&{}^3\nabla_s\, {}^3V^r = {}^3V^r{}_{|s} \equiv {}^3V^{(a)}_{|s}\, {}^3e^r_{(a)},\\
&\qquad \Rightarrow {}^3V^{(a)}{}_{|s} = \partial_s\, {}^3V^{(a)} + {}^3\omega^{(a)}_{s(b)}\, {}^3V^{(b)} = \partial_s\, {}^3V^{(a)} + \delta^{(a)(c)}\epsilon_{(c)(b)(d)}\, {}^3\omega_{s(d)}\, {}^3V^{(b)},\\
&{}^3\nabla_s\, {}^3V_r = {}^3V_{r|s} = {}^3V_{(a)|s}\, {}^3e^{(a)}_r,\\
&\qquad \Rightarrow {}^3V_{(a)|s} = \partial_s\, {}^3V_{(a)} - {}^3V_{(b)}\, {}^3\omega^{(b)}_{s(a)} = \partial_s\, {}^3V_{(a)} - {}^3V_{(b)}\delta^{(b)(c)}\, \epsilon_{(c)(a)(d)}\, {}^3\omega_{s(d)}.
\end{aligned} \tag{35}
$$

For the field strength and the curvature tensors we have

$$
\begin{aligned}
{}^3\Omega^{(a)}{}_{(b)(c)(d)} &= {}^3e_{(c)}({}^3\omega^{(a)}_{(d)(b)}) - {}^3e_{(d)}({}^3\omega^{(a)}_{(c)(b)}) +\\
&+ {}^3\omega^{(n)}_{(d)(b)}\, {}^3\omega^{(a)}_{(c)(n)} - {}^3\omega^{(n)}_{(c)(b)}\, {}^3\omega^{(a)}_{(d)(n)} - ({}^3\omega^{(n)}_{(c)(d)} - {}^3\omega^{(n)}_{(d)(c)})^3\omega^{(a)}_{(a)(b)} =\\
&= {}^3e^{(a)}_r\, {}^3R^r{}_{stw}\, {}^3e^s_{(b)}\, {}^3e^t_{(c)}\, {}^3e^w_{(d)},
\end{aligned}
$$

$$
\begin{aligned}
{}^3\Omega_{rs}{}^{(a)}{}_{(b)} &= {}^3e^{(c)}_r\, {}^3e^{(d)}_s\, {}^3\Omega^{(a)}{}_{(b)(c)(d)} = {}^3R^t{}_{wrs}\, {}^3e^{(a)}_t\, {}^3e^w_{(b)} =\\
&= \partial_r\, {}^3\omega^{(a)}_{s(b)} - \partial_s\, {}^3\omega^{(a)}_{r(b)} + {}^3\omega^{(a)}_{r(c)}\, {}^3\omega^{(c)}_{s(b)} - {}^3\omega^{(a)}_{s(c)}\, {}^3\omega^{(c)}_{r(b)} =\\
&= \delta^{(a)(c)}\, {}^3\Omega_{rs(c)(b)} = \delta^{(a)(c)}\, \epsilon_{(c)(b)(d)}\, {}^3\Omega_{rs(d)},
\end{aligned}
$$

$$
{}^3\Omega_{rs(a)} = \frac{1}{2}\epsilon_{(a)(b)(c)}\, {}^3\Omega_{rs(b)(c)} = \partial_r\, {}^3\omega_{s(a)} - \partial_s\, {}^3\omega_{r(a)} - \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(b)}\, {}^3\omega_{s(c)},
$$

$$
\begin{aligned}
{}^3R^r{}_{stw} &= \epsilon_{(a)(b)(c)}\, {}^3e^r_{(a)}\, \delta_{(b)(n)}\, {}^3e^{(n)}_s\, {}^3\Omega_{tw(c)},\\
{}^3R_{rs} &= \epsilon_{(a)(b)(c)}\, {}^3e^u_{(a)}\, \delta_{(b)(n)}\, {}^3e^{(n)}_r\, {}^3\Omega_{us(c)},\\
{}^3R &= \epsilon_{(a)(b)(c)}\, {}^3e^r_{(a)}\, {}^3e^s_{(b)}\, {}^3\Omega_{rs(c)}.
\end{aligned}
\tag{36}
$$

The first Bianchi identity (4) ${}^3R^t{}_{rsu} + {}^3R^t{}_{sur} + {}^3R^t{}_{urs} \equiv 0$ implies the cyclic identity ${}^3\Omega_{rs(a)}\, {}^3e^s_{(a)} \equiv 0$.

Under local SO(3) rotations R $[R^{-1} = R^t]$ we have

$$
\begin{aligned}
{}^3\omega^{(a)}_{r(b)} &\mapsto [R\, {}^3\omega_r\, R^T - R\partial_r\, R^T]^{(a)}{}_{(b)},\\
{}^3\Omega_{rs}{}^{(a)}{}_{(b)} &\mapsto [R\, {}^3\Omega_{rs}\, R^T]^{(a)}{}_{(b)}.
\end{aligned}
\tag{37}
$$

Since the flat metric $\delta_{(a)(b)}$ has signature $(+++)$, we have ${}^3V^{(a)} = \delta^{(a)(b)}\, {}^3V_{(b)} = {}^3V_{(a)}$ and one can simplify the notations by using only lower (a) indices $[{}^3e^{(a)}_r = {}^3e_{(a)r}]$. For instance, we have

$$
\begin{aligned}
{}^3\Gamma^u_{rs} &= {}^3\Gamma^u_{sr} = \frac{1}{2}\, {}^3e^u_{(a)}\Big[\partial_r\, {}^3e_{(a)s} + \partial_s\, {}^3e_{(a)r} +\\
&+ {}^3e^v_{(a)}\Big({}^3e_{(b)r}(\partial_s\, {}^3e_{(b)v} - \partial_v\, {}^3e_{(b)s}) + {}^3e_{(b)s}(\partial_r\, {}^3e_{(b)v} - \partial_v\, {}^3e_{(b)r})\Big)\Big] =\\
&= \frac{1}{2}\epsilon_{(a)(b)(c)}\, {}^3e^u_{(a)}({}^3e_{(b)r}\, {}^3\omega_{s(c)} + {}^3e_{(b)s}\, {}^3\omega_{r(c)}) - \frac{1}{2}({}^3e_{(a)r}\partial_s\, {}^3e^u_{(a)} + {}^3e_{(a)s}\partial_r\, {}^3e^u_{(a)}),\\
{}^3\omega_{r(a)(b)} &= -{}^3\omega_{r(b)(a)} = \frac{1}{2}\Big[{}^3e^s_{(a)}(\partial_r\, {}^3e_{(b)s} - \partial_s\, {}^3e_{(b)r}) +\\
&+ {}^3e^s_{(b)}(\partial_s\, {}^3e_{(a)r} - \partial_r\, {}^3e_{(a)s}) + {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^3e_{(c)r}(\partial_v\, {}^3e_{(c)u} - \partial_u\, {}^3e_{(c)v})\Big] =\\
&= \frac{1}{2}\Big[{}^3e_{(a)u}\partial_r\, {}^3e^u_{(b)} - {}^3e_{(b)u}\partial_r\, {}^3e^u_{(a)} + {}^3\Gamma^u_{rs}({}^3e_{(a)u}\, {}^3e^s_{(b)} - {}^3e_{(b)u}\, {}^3e^s_{(a)})\Big],\\
{}^3\omega_{r(a)} &= \frac{1}{2}\epsilon_{(a)(b)(c)}\Big[{}^3e^u_{(b)}(\partial_r\, {}^3e_{(c)u} - \partial_u\, {}^3e_{(c)r}) +\\
&+ \frac{1}{2}\, {}^3e^u_{(b)}\, {}^3e^v_{(c)}\, {}^3e_{(d)r}(\partial_v\, {}^3e_{(d)u} - \partial_u\, {}^3e_{(d)v})\Big],\\
{}^3\Omega_{rs(a)} &= \frac{1}{2}\epsilon_{(a)(b)(c)}\Big[\partial_r\, {}^3e^u_{(b)}\partial_s\, {}^3e_{(c)u} - \partial_s\, {}^3e^u_{(b)}\partial_r\, {}^3e_{(c)u} +\\
&+ {}^3e^u_{(b)}(\partial_u\partial_s\, {}^3e_{(c)r} - \partial_u\partial_r\, {}^3e_{(c)s}) +\\
&+ \frac{1}{2}\Big({}^3e^u_{(b)}\, {}^3e^v_{(c)}(\partial_r\, {}^3e_{(d)s} - \partial_s\, {}^3e_{(d)r})(\partial_v\, {}^3e_{(d)u} - \partial_u\, {}^3e_{(d)v}) +\\
&+ ({}^3e_{(d)s}\partial_r - {}^3e_{(d)r}\partial_s)[{}^3e^u_{(b)}\, {}^3e^v_{(c)}(\partial_v\, {}^3e_{(d)u} - \partial_u\, {}^3e_{(d)v})]\Big)\Big] -\\
&- \frac{1}{8}[\delta_{(a)(b_1)}\epsilon_{(c_1)(c_2)(b_2)} + \delta_{(a)(b_2)}\epsilon_{(c_1)(c_2)(b_1)} + \delta_{(a)(c_1)}\epsilon_{(b_1)(b_2)(c_2)} + \delta_{(a)(c_2)}\epsilon_{(b_1)(b_2)(c_1)}] \times\\
&\quad {}^3e^{u_1}_{(b_1)}\, {}^3e^{u_2}_{(b_2)}\Big[(\partial_r\, {}^3e_{(c_1)u_1} - \partial_{u_1}\, {}^3e_{(c_1)r})(\partial_s\, {}^3e_{(c_2)u_2} - \partial_{u_2}\, {}^3e_{(c_2)s}) +\\
&+ \frac{1}{2}\Big({}^3e^{v_2}_{(c_2)}\, {}^3e_{(d)s}(\partial_r\, {}^3e_{(c_1)u_1} - \partial_{u_1}\, {}^3e_{(c_1)r})(\partial_{v_2}\, {}^3e_{(d)u_2} - \partial_{u_2}\, {}^3e_{(d)v_2}) +
\end{aligned}
$$

$$+\,{}^3e^{v_1}_{(c_1)}\,{}^3e_{(d)r}(\partial_s\,{}^3e_{(c_2)u_2}-\partial_{u_2}\,{}^3e_{(c_2)s})(\partial_{v_1}\,{}^3e_{(d)u_1}-\partial_{u_1}\,{}^3e_{(d)v_1})\Big)+$$
$$+\,\frac{1}{4}\,{}^3e^{v_1}_{(c_1)}\,{}^3e^{v_2}_{(c_2)}\,{}^3e_{(d_1)r}\,{}^3e_{(d_2)s}(\partial_{v_1}\,{}^3e_{(d_1)u_1}-\partial_{u_1}\,{}^3e_{(d_1)v_1})(\partial_{v_2}\,{}^3e_{(d_2)u_2}-\partial_{u_2}\,{}^3e_{(d_2)v_2})\Big],$$
$$\begin{aligned}{}^3\Omega_{rs(a)(b)} &= \epsilon_{(a)(b)(c)}\,{}^3\Omega_{rs(c)},\\ {}^3R_{rsuv} &= \epsilon_{(a)(b)(c)}\,{}^3e_{(a)r}\,{}^3e_{(b)s}\,{}^3\Omega_{uv(c)},\\ {}^3R_{rs} &= \frac{1}{2}\epsilon_{(a)(b)(c)}\,{}^3e^u_{(a)}\Big[{}^3e_{(b)r}\,{}^3\Omega_{us(c)}+{}^3e_{(b)s}\,{}^3\Omega_{ur(c)}\Big],\\ {}^3R &= \epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,{}^3e^s_{(b)}\,{}^3\Omega_{rs(c)}.\end{aligned} \tag{38}$$

In the family of Σ_τ-adapted frames and coframes on M^4, we can select special tetrads and cotetrads ${}^4_{(\Sigma)}\check{E}_{(\alpha)}$ and ${}^4_{(\Sigma)}\check{\theta}^{(\alpha)}$ also adapted to a given set of triads and cotriads on Σ_τ

$$\begin{aligned}&{}^4_{(\Sigma)}\check{E}^\mu_{(\alpha)}=\{{}^4_{(\Sigma)}\check{E}^\mu_{(o)}=l^\mu=\hat{b}^\mu_l=\frac{1}{N}(b^\mu_\tau-N^r b^\mu_r);\ {}^4_{(\Sigma)}\check{E}^\mu_{(a)}={}^3e^s_{(a)}b^\mu_s\},\\ &{}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu=\{{}^4_{(\Sigma)}\check{E}^{(o)}_\mu=\epsilon l_\mu=\hat{b}^l_\mu=N b^\tau_\mu;\ {}^4_{(\Sigma)}\check{E}^{(a)}_\mu={}^3e^{(a)}_s\hat{b}^s_\mu\},\\ &{}^4_{(\Sigma)}\check{E}^\mu_{(\alpha)}\quad {}^4g_{\mu\nu}\ {}^4_{(\Sigma)}\check{E}^\nu_{(\beta)}={}^4\eta_{(\alpha)(\beta)},\end{aligned} \tag{39}$$

where b^μ_r and b^r_μ are defined in Eqs.(6). The components of these tetrads and cotetrads in the holonomic bases are ([59,66]; ${}^4_{(\Sigma)}\check{\check{E}}^{(o)}_r=0$ is the Schwinger time gauge condition [60])

$$\begin{aligned}{}^4_{(\Sigma)}\check{\check{E}}^A_{(\alpha)} &= {}^4_{(\Sigma)}\check{E}^\mu_{(\alpha)}\,b^A_\mu,\quad \Rightarrow {}^4_{(\Sigma)}\check{\check{E}}^A_{(o)}=\epsilon l^A,\\ &{}^4_{(\Sigma)}\check{\check{E}}^\tau_{(o)}=\frac{1}{N},\qquad {}^4_{(\Sigma)}\check{\check{E}}^\tau_{(a)}=0,\\ &{}^4_{(\Sigma)}\check{\check{E}}^r_{(o)}=-\frac{N^r}{N},\qquad {}^4_{(\Sigma)}\check{\check{E}}^r_{(a)}={}^3e^r_{(a)};\\ {}^4_{(\Sigma)}\check{\check{E}}^{(\alpha)}_A &= {}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu\,b^\mu_A,\quad \Rightarrow {}^4_{(\Sigma)}\check{\check{E}}^{(o)}_A=l_A,\\ &{}^4_{(\Sigma)}\check{\check{E}}^{(o)}_\tau=N,\qquad {}^4_{(\Sigma)}\check{\check{E}}^{(a)}_\tau=N^r\,{}^3e^{(a)}_r=N^{(a)},\\ &{}^4_{(\Sigma)}\check{\check{E}}^{(o)}_r=0,\qquad {}^4_{(\Sigma)}\check{\check{E}}^{(a)}_r={}^3e^{(a)}_r,\\ &{}^4_{(\Sigma)}\check{E}^A_{(\alpha)}\,{}^4g_{AB}\,{}^4_{(\Sigma)}\check{E}^B_{(\beta)}={}^4\eta_{(\alpha)(\beta)}.\end{aligned} \tag{40}$$

With the cotetrads ${}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\sigma))$ we can build the vector $\overset{\circ}{V}{}^{(\alpha)}=l^\mu(z(\sigma))\,{}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\sigma))=(1;\vec{0})$: it is the same unit timelike future-pointing Minkowski 4-vector in the tangent plane of each point $z^\mu(\sigma)=z^\mu(\tau,\vec{\sigma})\in\Sigma_\tau\subset M^4$ for every τ and $\vec{\sigma}$; we have $\overset{\circ}{V}{}^{(\alpha)}\ {}^4\eta_{(\alpha)(\beta)}\,\overset{\circ}{V}{}^{(\beta)}=\epsilon$.

Let ${}^4E^\mu_{(\alpha)}(z)$ and ${}^4E^{(\alpha)}_\mu(z)$ be arbitrary tetrads and cotetrads on M^4. Let us define the point-dependent Minkowski 4-vector $V^{(\alpha)}(z(\sigma))=l^\mu(z(\sigma))\,{}^4E^{(\alpha)}_\mu(z(\sigma))$ (assumed to be future-pointing), which satisfies $V^{(\alpha)}(z(\sigma))\,{}^4\eta_{(\alpha)(\beta)}\,V^{(\beta)}(z(\sigma))=\epsilon$, so that $V^{(\alpha)}(z(\sigma))=(V^{(o)}(z(\sigma))=+\sqrt{1+\sum_r V^{(r)2}(z(\sigma))};V^{(r)}(z(\sigma))\overset{def}{=}\varphi^{(r)}(\sigma))$: therefore, the point-dependent Minkowski 4-vector $V^{(\alpha)}(z(\sigma))$ depends only on the three functions $\varphi^{(r)}(\sigma)$ [one has $\varphi^{(r)}(\sigma)=$

$-\epsilon\varphi_{(r)}(\sigma)$ since ${}^4\eta_{rs} = -\epsilon\,\delta_{rs}$; having the Euclidean signature (+++) for both $\epsilon = \pm 1$, we shall define the Kronecker delta as $\delta^{(i)(j)} = \delta^{(i)}_{(j)} = \delta_{(i)(j)}$]. If we introduce the point-dependent Lorentz transformation

$$
\begin{aligned}
L^{(\alpha)}{}_{(\beta)}(V(z(\sigma)); \overset{\circ}{V}) &= \delta^{(\alpha)}_{(\beta)} + 2\epsilon V^{(\alpha)}(z(\sigma))\overset{\circ}{V}_{(\beta)} - \epsilon \frac{(V^{(\alpha)}(z(\sigma)) + \overset{\circ}{V}{}^{(\alpha)})(V_{(\beta)}(z(\sigma)) + \overset{\circ}{V}_{(\beta)})}{1 + V^{(o)}(z(\sigma))} = \\
&= \begin{pmatrix} V^{(o)} & -\epsilon V_{(j)} \\ V^{(i)} & \delta^{(i)}_{(j)} - \epsilon \frac{V^{(i)}V_{(j)}}{1+V^{(o)}} \end{pmatrix} (z(\sigma)),
\end{aligned}
\tag{41}
$$

which is the standard Wigner boost for timelike Poincaré orbits [see Ref. [83]], one has by construction

$$
V^{(\alpha)}(z(\sigma)) = l^\mu(z(\sigma))\, {}^4E^{(\alpha)}_\mu(z(\sigma)) = L^{(\alpha)}{}_{(\beta)}(V(z(\sigma)); \overset{\circ}{V})\, \overset{\circ}{V}{}^{(\beta)}. \tag{42}
$$

Therefore, we shall define an arbitrary cotretad ${}^4E^{(\alpha)}_\mu(z(\sigma))$ on M^4 starting from the special Σ_τ- and cotriad-adapted cotetrad ${}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\sigma))$ by means of the formula

$$
{}^4E^{(\alpha)}_\mu(z(\sigma)) = L^{(\alpha)}{}_{(\beta)}(V(z(\sigma)); \overset{\circ}{V})\, {}^4_{(\Sigma)}\check{E}^{(\beta)}_\mu(z(\sigma)). \tag{43}
$$

Let us remark that with this definition we are putting equal to zero, by convention, the angles of an arbitrary 3-rotation of $b^s_\mu(z(\sigma))$ [i.e. of the choice of the three axes tangent to Σ_τ] inside ${}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\sigma))$.

Since $\varphi^{(a)}(\sigma) = V^{(a)}(z(\sigma)) = l^\mu(z(\sigma))\, {}^4E^{(a)}_\mu(z(\sigma))$ are the three parameters of the Wigner boost [$\varphi^{(a)} = \bar{\gamma}\beta^{(a)}$, $\bar{\gamma} = \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}$, $\beta^{(a)} = \varphi^{(a)}/\sqrt{1 + \sum_{(c)} \varphi^{(c)2}}$], the previous equation can be rewritten in the following form [remembering that $\varphi^{(a)} = -\epsilon\varphi_{(a)}$]

$$
\begin{pmatrix} {}^4E^{(o)}_\mu \\ {}^4E^{(a)}_\mu \end{pmatrix} (z(\sigma)) = \begin{pmatrix} \sqrt{1 + \sum_{(c)} \varphi^{(c)2}} & -\epsilon\varphi_{(b)} \\ \varphi^{(a)} & \delta^{(a)}_{(b)} - \epsilon \frac{\varphi^{(a)}\varphi_{(b)}}{1+\sqrt{1+\sum_{(c)} \varphi^{(c)2}}} \end{pmatrix} (z(\sigma)) \begin{pmatrix} l_\mu \\ {}^3e^{(b)}_s\, b^s_\mu \end{pmatrix} (\sigma). \tag{44}
$$

If we go to holonomic bases, ${}^4E^{(\alpha)}_A(z(\sigma)) = {}^4E^{(\alpha)}_\mu(z(\sigma))\, b^\mu_A(\sigma)$ and ${}^4_{(\Sigma)}\check{E}^{(\alpha)}_A(z(\sigma)) = {}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\sigma))\, b^\mu_A(\sigma)$, one has

$$
\begin{aligned}
\begin{pmatrix} {}^4E^{(o)}_A \\ {}^4E^{(a)}_A \end{pmatrix} &= \begin{pmatrix} \sqrt{1 + \sum_{(c)} \varphi^{(c)2}} & -\epsilon\varphi_{(b)} \\ \varphi^{(a)} & \delta^{(a)}_{(b)} - \epsilon \frac{\varphi^{(a)}\varphi_{(b)}}{1+\sqrt{1+\sum_{(c)} \varphi^{(c)2}}} \end{pmatrix} \times \\
&\begin{pmatrix} {}^4_{(\Sigma)}\check{E}^{(o)}_A = (N; \vec{0}) \\ {}^4_{(\Sigma)}\check{E}^{(b)}_A = (N^{(b)} = {}^3e^{(b)}_r N^r; {}^3e^{(b)}_r) \end{pmatrix},
\end{aligned}
\tag{45}
$$

so that we get that the cotetrad in holonomic basis can be expressed in terms of N, $N^{(a)} = {}^3e^{(a)}_s N^s = N_{(a)}$, $\varphi^{(a)}$ and ${}^3e^{(a)}_r$ [${}^3g_{rs} = \sum_{(a)} {}^3e_{(a)r}\, {}^3e_{(a)s}$]

$$
\begin{aligned}
&{}^4E^{(o)}_\tau(z(\sigma)) = \sqrt{1+\sum_{(c)}\varphi^{(c)2}(\sigma)}\, N(\sigma) + \sum_{(a)}\varphi^{(a)}(\sigma) N^{(a)}(\sigma),\\
&{}^4E^{(o)}_r(z(\sigma)) = \sum_{(a)}\varphi^{(a)}(\sigma)\, {}^3e^{(a)}_r(\sigma),\\
&{}^4E^{(a)}_\tau(z(\sigma)) = \varphi^{(a)}(\sigma)N(\sigma) + \sum_{(b)}[\delta^{(a)}_{(b)} - \epsilon\frac{\varphi^{(a)}(\sigma)\varphi_{(b)}(\sigma)}{1+\sqrt{1+\sum_{(c)}\varphi^{(c)2}(\sigma)}}]N^{(b)}(\sigma),\\
&{}^4E^{(a)}_r(z(\sigma)) = \sum_{(b)}[\delta^{(a)}_{(b)} - \epsilon\frac{\varphi^{(a)}(\sigma)\varphi_{(b)}(\sigma)}{1+\sqrt{1+\sum_{(c)}\varphi^{(c)2}(\sigma)}}]^3e^{(b)}_r(\sigma),\\
&\Rightarrow {}^4g_{AB} = {}^4E^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4E^{(\beta)}_B = {}^4_{(\Sigma)}\check{E}^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4_{(\Sigma)}\check{E}^{(\beta)}_B =\\
&= \epsilon \begin{pmatrix} (N^2 - {}^3g_{rs}N^rN^s) & -{}^3g_{st}N^t \\ -{}^3g_{rt}N^t & -{}^3g_{rs} \end{pmatrix},
\end{aligned}
\tag{46}
$$

with the last line in accord with Eqs.(6); we have used $L^{T4}\eta L = {}^4\eta$, valid for every Lorentz transformation. We find $L^{-1}(V,\overset{\circ}{V}) = {}^4\eta L^T(V,\overset{\circ}{V})^4\eta = L(V,\overset{\circ}{V})|_{\varphi^{(a)}\mapsto -\varphi^{(a)}}$ and $[{}^4E^A_{(\alpha)} = {}^4E^\mu_{(\alpha)}b^A_\mu,\ {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(\alpha)} = {}^4_{(\Sigma)}\check{E}^\mu_{(\alpha)}b^A_\mu]$

$$
\begin{aligned}
&\begin{pmatrix} {}^4E^\mu_{(o)} \\ {}^4E^\mu_{(a)} \end{pmatrix} = \begin{pmatrix} \sqrt{1+\sum_{(c)}\varphi^{(c)2}} & -\varphi^{(b)} \\ \epsilon\varphi_{(a)} & \delta^{(b)}_{(a)} - \epsilon\frac{\varphi_{(a)}\varphi^{(b)}}{1+\sqrt{1+\sum_{(c)}\varphi^{(c)2}}} \end{pmatrix} \begin{pmatrix} l^\mu \\ b^\mu_s\, {}^3e^s_{(b)} \end{pmatrix},\\
&\begin{pmatrix} {}^4E^A_{(o)} \\ {}^4E^A_{(a)} \end{pmatrix} = \begin{pmatrix} \sqrt{1+\sum_{(c)}\varphi^{(c)2}} & -\varphi^{(b)} \\ \epsilon\varphi^{(a)} & \delta^{(b)}_{(a)} - \epsilon\frac{\varphi_{(a)}\varphi^{(b)}}{1+\sqrt{1+\sum_{(c)}\varphi^{(c)2}}} \end{pmatrix} \begin{pmatrix} {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(o)} = (1/N; -N^r/N) \\ {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(b)} = (0; {}^3e^r_{(b)}) \end{pmatrix},\\
&{}^4E^\tau_{(o)}(z(\sigma)) = \sqrt{1+\sum_{(c)}\varphi^{(c)2}(\sigma)}\frac{1}{N(\sigma)},\\
&{}^4E^r_{(o)}(z(\sigma) = -\sqrt{1+\sum_{(c)}\varphi^{(c)2}(\sigma)}\frac{N^r(\sigma)}{N(\sigma)} - \varphi^{(b)}(\sigma)\, {}^3e^r_{(b)}(\sigma),\\
&{}^4E^\tau_{(a)}(z(\sigma)) = \epsilon\frac{\varphi_{(a)}(\sigma)}{N(\sigma)},\\
&{}^4E^r_{(a)}(z(\sigma)) = -\epsilon\varphi_{(a)}(\sigma)\frac{N^r(\sigma)}{N(\sigma)} + \sum_{(b)}[\delta^{(b)}_{(a)} - \epsilon\frac{\varphi_{(a)}(\sigma)\varphi^{(b)}(\sigma)}{1+\sqrt{1+\sum_{(c)}\varphi^{(c)2}(\sigma)}}]\, {}^3e^r_{(b)}(\sigma),\\
&\Rightarrow {}^4g^{AB} = {}^4E^A_{(\alpha)}\, {}^4\eta^{(\alpha)(\beta)}\, {}^4E^B_{(\beta)} = {}^4_{(\Sigma)}\check{E}^A_{(\alpha)}\, {}^4\eta_{(\alpha)(\beta)}\, {}^4_{(\Sigma)}\check{E}^B_{(\beta)} =\\
&= \epsilon \begin{pmatrix} \frac{1}{N^2} & -\frac{N^s}{N^2} \\ -\frac{N^r}{N^2} & -({}^3g^{rs} - \frac{N^rN^s}{N^2}) \end{pmatrix},
\end{aligned}
\tag{47}
$$

with the last line in accord with Eqs.(6).

From ${}^4_{(\Sigma)}\check{\tilde{E}}^{(\alpha)}_A(z(\sigma)) = (L^{-1})^{(\alpha)}{}_{(\beta)}(V(z(\sigma));\overset{\circ}{V})\, {}^4E^{(\beta)}_A(z(\sigma))$ and ${}^4_{(\Sigma)}\check{\tilde{E}}^A_{(\alpha)}(z(\sigma)) = {}^4E^A_{(\beta)}\, (L^{-1})^{(\beta)}{}_{(\alpha)}(V(z(\sigma));\overset{\circ}{V})$ it turns out [83] that the flat indices (a) of the adapted tetrads ${}^4_{(\Sigma)}\check{E}^\mu_{(a)}$ and of the triads ${}^3e^r_{(a)}$ and cotriads ${}^3e^{(a)}_r$ on Σ_τ transform as Wigner spin 1 indices under point-dependent SO(3) Wigner rotations $R^{(a)}{}_{(b)}(V(z(\sigma));\Lambda(z(\sigma)))$ associated with Lorentz transformations $\Lambda^{(\alpha)}{}_{(\beta)}(z)$ in the tangent plane to M^4 in the same point $[R^{(\alpha)}{}_{(\beta)}(\Lambda(z(\sigma));V(z(\sigma))) = [L(\overset{\circ}{V};V(z(\sigma)))\,\Lambda^{-1}(z(\sigma))\,L(\Lambda(z(\sigma))V(z(\sigma));\overset{\circ}{V})]^{(\alpha)}{}_{(\beta)} = \begin{pmatrix} 1 & 0 \\ 0 & R^{(a)}{}_{(b)}(V(z(\sigma));\Lambda(z(\sigma))) \end{pmatrix}]$. Instead the index (o) of the adapted tetrads ${}^4_{(\Sigma)}\check{E}^\mu_{(o)}$ is a local Lorentz scalar in each point. Therefore, the adapted tetrads in the holonomic basis should be denoted as ${}^4_{(\Sigma)}\check{\tilde{E}}^A_{(\bar{\alpha})}$, with $(\bar{o})$ and $A=(\tau,r)$ Lorentz scalar indices and with $(\bar{a})$ Wigner spin 1 indices; we shall go on with the indices $(o),(a)$ without the overbar for the sake of simplicity. In this way the tangent planes to Σ_τ in M^4 are described in a Wigner covariant way, reminiscent of the flat rest-frame covariant instant form of dynamics introduced in Minkowski spacetime in Ref. [11]. Similar conclusions are reached independently in Ref. [84] in the framework of nonlinear Poincaré gauge theory [the vector fields e_α and the 1-forms θ^α of that paper correspond to $X_{\tilde{A}}$ and $\theta^{\tilde{A}}$ in Eq.(8) respectively].

Therefore, an arbitrary tetrad field, namely a (in general nongeodesic) congruence of observers' timelike worldlines with 4-velocity field $u^A(\tau,\vec{\sigma}) = {}^4E^A_{(o)}(\tau,\vec{\sigma})$, can be obtained with a pointwise Wigner boost from the special surface-forming timelike congruence whose 4-velocity field is the normal to Σ_τ $l^A(\tau,\vec{\sigma}) = \epsilon\, {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(o)}(\tau,\vec{\sigma})$ [it is associated with the 3+1 splitting of M^4 with leaves Σ_τ; see Appendix A].

We can invert Eqs.(47) to get N, $N^r = {}^3e^r_{(a)}N^{(a)}$, $\varphi^{(a)}$ and ${}^3e^r_{(a)}$ in terms of the tetrads ${}^4E^A_{(\alpha)}$

$$\begin{aligned}
N &= \frac{1}{\sqrt{[{}^4E^\tau_{(o)}]^2 - \sum_{(c)}[{}^4E^\tau_{(c)}]^2}}.\\
N^r &= -\frac{{}^4E^\tau_{(o)}\,{}^4E^r_{(0)} - \sum_{(c)}{}^4E^\tau_{(c)}\,{}^4E^r_{(c)}}{[{}^4E^\tau_{(0)}]^2 - \sum_{(c)}[{}^4E^\tau_{(c)}]^2}\\
\varphi_{(a)} &= \frac{\epsilon\;{}^4E^\tau_{(a)}}{\sqrt{[{}^4E^\tau_{(o)}]^2 - \sum_{(c)}[{}^4E^\tau_{(c)}]^2}}\\
{}^3e^r_{(a)} &= \sum_{(b)} B_{(a)(b)}({}^4E^r_{(b)} + N^r\;{}^4E^\tau_{(b)})\\
&\qquad B_{(a)(b)} = \delta_{(a)(b)} - \frac{{}^4E^\tau_{(a)}\,{}^4E^\tau_{(b)}}{{}^4E^\tau_{(0)}[{}^4E^\tau_{(0)} + \sqrt{[{}^4E^\tau_{(0)}]^2 - \sum_{(c)}[{}^4E^\tau_{(c)}]^2}]}.
\end{aligned} \tag{48}$$

If ${}^3e^{-1} = det\,({}^3e^r_{(a)})$, then from the orthonormality condition we get ${}^3e_{(a)r} = {}^3e({}^3e^s_{(b)}\,{}^3e^t_{(c)} - {}^3e^t_{(b)}\,{}^3e^s_{(c)})$ [with $(a),(b),(c)$ and r,s,t cyclic] and it allows to express the cotriads in terms of the tetrads ${}^4E^A_{(\alpha)}$. Therefore, given the tetrads ${}^4E^A_{(\alpha)}$ [or equivalently the cotetrads ${}^4E^{(\alpha)}_A$] on M^4, an equivalent set of variables with the local Lorentz covariance replaced with local Wigner covariance are the lapse N, the shifts $N^{(a)} = N_{(a)} = {}^3e_{(a)r}N^r$, the Wigner-boost parameters $\varphi^{(a)} = -\epsilon\varphi_{(a)}$ and either the triads ${}^3e^r_{(a)}$ or the cotriads ${}^3e_{(a)r}$.

In Appendix A there is the expression in terms of the variables N, $N_{(a)}$, $\varphi_{(a)}$ and ${}^3e_{(a)r}$ [and/or ${}^3e^r_{(a)}$] of the connection coefficients ${}^4\Gamma^B_{AC}$, of the spin connection ${}^4\omega_{A(\alpha)(\beta)}$, of the field strength ${}^4\Omega_{AB(\alpha)(\beta)}$, of the Riemann tensor ${}^4R^A{}_{BCD}$ and of the Weyl tensor ${}^4C^A{}_{BCD}$ in the Σ_τ-adapted holonomic coordinate basis, where the 4-metric is ${}^4g_{AB}$. These formulas give the bridge to the reconstruction of the spacetime M^4 starting from the ADM tetrad description and show explicitly the dependence of 4-tensors on the undetermined lapse and shift functions.

IV. THE LAGRANGIAN AND THE HAMILTONIAN IN THE NEW VARIABLES.

Let us consider the ADM action (25) S_{ADM}; its independent variables in metric gravity have now the following expression in terms of N, $N^{(a)} = N_{(a)} = {}^3e^r_{(a)} N_r$, $\varphi^{(a)} = -\epsilon\varphi_{(a)}$, ${}^3e^{(a)}_r = {}^3e_{(a)r}$ $[\gamma = det\,({}^3g_{rs}) = ({}^3e)^2 = (det\,(e_{(a)r}))^2]$

$$\begin{aligned} &N, \qquad\qquad N_r = {}^3e^{(a)}_r N_{(a)} = {}^3e_{(a)r} N_{(a)},\\ &{}^3g_{rs} = {}^3e^{(a)}_r\, \delta_{(a)(b)}\, {}^3e^{(b)}_s = {}^3e_{(a)r}\, {}^3e_{(a)s}, \end{aligned} \tag{49}$$

so that the line element of M^4 becomes

$$ds^2 = \epsilon(N^2 - N_{(a)}N_{(a)})(d\tau)^2 - 2\epsilon N_{(a)}\, {}^3e_{(a)r} d\tau d\sigma^r - \epsilon\, {}^3e_{(a)r}\, {}^3e_{(a)s} d\sigma^r d\sigma^s = \epsilon\Big[N^2(d\tau)^2 - ({}^3e_{(a)r} d\sigma^r + N_{(a)} d\tau)({}^3e_{(a)s} d\sigma^s + N_{(a)} d\tau)\Big].$$

The extrinsic curvature takes the form $[N_{(a)|r} = {}^3e^s_{(a)} N_{s|r} = \partial_r N_{(a)} - \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(b)} N_{(c)}$ from Eq.(35)]

$$\begin{aligned} {}^3K_{rs} &= \hat b^\mu_r \hat b^\nu_s {}^3K_{\mu\nu} = \frac{1}{2N}(N_{r|s} + N_{s|r} - \partial_\tau\, {}^3g_{rs}) =\\ &= \frac{1}{2N}({}^3e_{(a)r}\delta^w_s + {}^3e_{(a)s}\delta^w_r)(N_{(a)|w} - \partial_\tau\, {}^3e_{(a)w}),\\ {}^3K_{r(a)} &= {}^3K_{rs}\, {}^3e^s_{(a)} = \frac{1}{2N}(\delta_{(a)(b)}\delta^w_r + {}^3e^w_{(a)}\, {}^3e_{(b)r})(N_{(b)|w} - \partial_\tau\, {}^3e_{(c)w}),\\ {}^3K &= \frac{1}{N}\, {}^3e^r_{(a)}(N_{(a)|r} - \partial_\tau\, {}^3e_{(a)r}), \end{aligned} \tag{50}$$

so that the ADM action in the new variables is

$$\begin{aligned} \hat S_{ADMT} &= \int d\tau \hat L_{ADMT} =\\ &= -\epsilon k \int d\tau d^3\sigma \{N\, {}^3e\, \epsilon_{(a)(b)(c)}\, {}^3e^r_{(a)}\, {}^3e^s_{(b)}\, {}^3\Omega_{rs(c)} +\\ &+ \frac{{}^3e}{2N}({}^3G_o^{-1})_{(a)(b)(c)(d)}\, {}^3e^r_{(b)}(N_{(a)|r} - \partial_\tau\, {}^3e_{(a)r})\, {}^3e^s_{(d)}(N_{(c)|s} - \partial_\tau\, {}^3e_{(c)\ s})\}, \end{aligned} \tag{51}$$

where we introduced the flat (with lower indices) inverse Wheeler-DeWitt supermetric

$$({}^3G_o^{-1})_{(a)(b)(c)(d)} = \delta_{(a)(c)}\delta_{(b)(d)} + \delta_{(a)(d)}\delta_{(b)(c)} - 2\delta_{(a)(b)}\delta_{(c)(d)}. \tag{52}$$

The flat supermetric is

$$\begin{aligned} {}^3G_{o(a)(b)(c)(d)} &= {}^3G_{o(b)(a)(c)(d)} = {}^3G_{o(a)(b)(d)(c)} = {}^3G_{o(c)(d)(a)(b)} =\\ &= \delta_{(a)(c)}\delta_{(b)(d)} + \delta_{(a)(d)}\delta_{(b)(c)} - \delta_{(a)(b)}\delta_{(c)(d)}, \end{aligned}$$

$$\frac{1}{2}\, {}^3G_{o(a)(b)(e)(f)}\, \frac{1}{2}\, {}^3G^{-1}_{o(e)(f)(c)(d)} = \frac{1}{2}[\delta_{(a)(c)}\delta_{(b)(d)} + \delta_{(a)(d)}\delta_{(b)(c)}]. \tag{53}$$

The new action does not depend on the 3 boost variables $\varphi^{(a)}$ [like the Higgs model Lagrangian in the unitary gauge does not depend on some of the Higgs fields [6,7]], contains lapse N and modified shifts $N_{(a)}$ as Lagrange multipliers, and is a functional independent from the second time derivatives of the fields. The canonical momenta and the Poisson brackets are

$$
\begin{aligned}
&\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) = \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau \varphi_{(a)}(\tau,\vec{\sigma})} = 0,\\
&\tilde{\pi}^{N}(\tau,\vec{\sigma}) = \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau N(\tau,\vec{\sigma})} = 0,\\
&\tilde{\pi}^{\vec{N}}_{(a)}(\tau,\vec{\sigma}) = \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau N_{(a)}(\tau,\vec{\sigma})} = 0,\\
&{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) = \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau {}^3e_{(a)r}(\tau,\vec{\sigma})} = [\frac{\epsilon k {}^3e}{N} ({}^3G^{-1}_o)_{(a)(b)(c)(d)}\, {}^3e^r_{(b)}\, {}^3e^s_{(d)}\, (N_{(c)|s} - \partial_\tau\, {}^3e_{(c)s})](\tau,\vec{\sigma}) =\\
&= 2\epsilon k [{}^3e ({}^3K^{rs} - {}^3e^r_{(c)}\, {}^3e^s_{(c)}\, {}^3K) {}^3e_{(a)s}](\tau,\vec{\sigma}),\\
&\\
&\{N(\tau,\vec{\sigma}), \tilde{\pi}^N(\tau,\vec{\sigma}^{'})\} = \delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\{N_{(a)}(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{N}}_{(b)}(\tau,\vec{\sigma}^{'})\} = \delta_{(a)(b)} \delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\{\varphi_{(a)}(\tau,\vec{\sigma}), \tilde{\pi}^{\vec{\varphi}}_{(b)}(\tau,\vec{\sigma}^{'})\} = \delta_{(a)(b)} \delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\{{}^3e_{(a)r}(\tau,\vec{\sigma}), {}^3\tilde{\pi}^s_{(b)}(\tau,\vec{\sigma}^{'})\} = \delta_{(a)(b)} \delta^s_r \delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\\
&\{{}^3e^r_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{\pi}^s_{(b)}(\tau,\vec{\sigma}^{'})\} = -{}^3e^r_{(b)}(\tau,\vec{\sigma})\, {}^3e^s_{(a)}(\tau,\vec{\sigma}) \delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\{{}^3e(\tau,\vec{\sigma}), {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}^{'})\} = {}^3e(\tau,\vec{\sigma})\, {}^3e^r_{(a)}(\tau,\vec{\sigma})\, \delta^3(\vec{\sigma},\vec{\sigma}^{'}),
\end{aligned}
\tag{54}
$$

where the Dirac delta distribution is a density of weight -1 [it behaves as $\sqrt{\gamma(\tau,\vec{\sigma})}$], because we have the $\vec{\sigma}^{'}$-reparametrization invariant result $\int d^3\sigma^{'} \delta^3(\vec{\sigma},\vec{\sigma}^{'}) f(\vec{\sigma}^{'}) = f(\vec{\sigma})$]. The momentum ${}^3\tilde{\pi}^r_{(a)}$ is a density of weight -1.

Besides the seven primary constraints $\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) \approx 0$, $\tilde{\pi}^N(\tau,\vec{\sigma}) \approx 0$, $\tilde{\pi}^{\vec{N}}_{(a)}(\tau,\vec{\sigma}) \approx 0$, there are the following three primary constraints (the generators of the inner rotations)

$$
\begin{aligned}
{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) &= \epsilon_{(a)(b)(c)}\, {}^3e_{(b)r}(\tau,\vec{\sigma})\, {}^3\tilde{\pi}^r_{(c)}(\tau,\vec{\sigma}) = \frac{1}{2} \epsilon_{(a)(b)(c)}\, {}^3\tilde{M}_{(b)(c)}(\tau,\vec{\sigma}) \approx 0,\\
&\Rightarrow {}^3\tilde{M}_{(a)(b)}(\tau,\vec{\sigma}) = \epsilon_{(a)(b)(c)}\, {}^3\tilde{M}_{(c)}(\tau,\vec{\sigma}) =\\
&= {}^3e_{(a)r}(\tau,\vec{\sigma})\, {}^3\tilde{\pi}^r_{(b)}(\tau,\vec{\sigma}) - {}^3e_{(b)r}(\tau,\vec{\sigma})\, {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) \approx 0.
\end{aligned}
\tag{55}
$$

By using Eqs.(53) and (54) we get the following inversion

$$
\begin{aligned}
{}^3e^r_{(a)}\ &(\ N_{(b)|r} - \partial_\tau\, {}^3e_{(b)r}) + {}^3e^r_{(b)} (N_{(a)|r} - \partial_\tau\, {}^3e_{(a)r}) =\\
&= \frac{\epsilon N}{2k\, {}^3e}\, {}^3G_{o(a)(b)(c)(d)}\, {}^3e_{(c)r}\, {}^3\tilde{\pi}^r_{(d)},
\end{aligned}
\tag{56}
$$

so that, even if this equation cannot be solved for $\partial_\tau\, {}^3e_{(a)r}$ [due to the degeneracy associated with the first class constraints], we can get the phase space expression of the extrinsic curvature without using the Hamilton equations

$$
\begin{aligned}
{}^3K_{rs} &= \frac{\epsilon}{4k\,{}^3e}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}\,{}^3e_{(b)s}\,{}^3e_{(c)u}\,{}^3\tilde{\pi}^u_{(d)},\\
{}^3K &= -\frac{\epsilon}{2k\sqrt{\gamma}}\,{}^3\tilde{\Pi} = -\frac{\epsilon}{4k\,{}^3e}\,{}^3e_{(a)r}\,{}^3\tilde{\pi}^r_{(a)}.
\end{aligned}
\tag{57}
$$

Since at the Lagrangian level the primary constraints are identically zero, we have

$$
\begin{aligned}
{}^3\tilde{\pi}^r_{(a)} &= {}^3e^r_{(b)}\,{}^3e_{(b)s}\,{}^3\tilde{\pi}^s_{(a)} = \frac{1}{2}{}^3e^r_{(b)}[{}^3e_{(a)s}\,{}^3\tilde{\pi}^s_{(b)} + {}^3e_{(b)s}\,{}^3\tilde{\pi}^s_{(a)}] - \frac{1}{2}{}^3\tilde{M}_{(a)(b)}\,{}^3e^r_{(b)} \equiv\\
&\equiv \frac{1}{2}{}^3e^r_{(b)}[{}^3e_{(a)s}\,{}^3\tilde{\pi}^s_{(b)} + {}^3e_{(b)s}\,{}^3\tilde{\pi}^s_{(a)}],\\
{}^3\tilde{\pi}^r_{(a)}\,\partial_\tau\,{}^3e_{(a)r} &\equiv \frac{1}{2}[{}^3e_{(a)s}\,{}^3\tilde{\pi}^s_{(b)} + {}^3e_{(b)s}\,{}^3\tilde{\pi}^s_{(a)}]{}^3e^r_{(b)}\,\partial_\tau\,{}^3e_{(a)r} \equiv\\
&\equiv {}^3\tilde{\pi}^r_{(a)}\,N_{(a)|r} - \frac{N}{4k\,{}^3e}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)s}\,{}^3\tilde{\pi}^s_{(b)}\,{}^3e_{(c)r}\,{}^3\tilde{\pi}^r_{(d)},
\end{aligned}
\tag{58}
$$

and the canonical Hamiltonian is

$$
\begin{aligned}
\hat{H}_{(c)} &= \int d^3\sigma[\tilde{\pi}^N\partial_\tau N + \tilde{\pi}^{\vec{N}}_{(a)}\partial_\tau N_{(a)} + \tilde{\pi}^{\vec{\varphi}}_{(a)}\partial_\tau\varphi_{(a)} + {}^3\tilde{\pi}^r_{(a)}\partial_\tau{}^3e_{(a)r}](\tau,\vec{\sigma}) - \hat{L}_{ADMT} =\\
&= \int_{\Sigma_\tau} d^3\sigma[\epsilon N\,(k\,{}^3e\,\epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,{}^3e^s_{(b)}\,{}^3\Omega_{rs(c)} -\\
&- \frac{1}{8k\,{}^3e}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}\,{}^3\tilde{\pi}^r_{(b)}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)}) -\\
&- N_{(a)}\,{}^3\tilde{\pi}^r_{(a)|r}](\tau,\vec{\sigma}) + \int_{\partial\Sigma_\tau} d^2\Sigma_r[N_{(a)}\,{}^3\tilde{\pi}^r_{(a)}](\tau,\vec{\sigma}).
\end{aligned}
\tag{59}
$$

In this paper we shall ignore the surface term.

The Dirac Hamiltonian is

$$
\hat{H}_{(D)} = \hat{H}_{(c)} + \int d^3\sigma[\lambda_N\,\tilde{\pi}^N + \lambda^{\vec{N}}_{(a)}\,\tilde{\pi}^{\vec{N}}_{(a)} + \lambda^{\vec{\varphi}}_{(a)}\,\tilde{\pi}^{\vec{\varphi}}_{(a)} + \mu_{(a)}\,{}^3\tilde{M}_{(a)}](\tau,\vec{\sigma}).
\tag{60}
$$

The τ-constancy of the ten primary constraints generates four secondary constraints [from $\partial_\tau\,\tilde{\pi}^N(\tau,\vec{\sigma}) \approx 0$ and from $\partial_\tau\,\tilde{\pi}^{\vec{N}}_{(a)}(\tau,\vec{\sigma}) \approx 0$]

$$
\begin{aligned}
\hat{\mathcal{H}}(\tau,\vec{\sigma}) &= \epsilon\Big[k\,{}^3e\,\epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,{}^3e^s_{(b)}\,{}^3\Omega_{rs(c)} -\\
&- \frac{1}{8k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}\,{}^3\tilde{\pi}^r_{(b)}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)}\Big](\tau,\vec{\sigma}) =\\
&= \epsilon\Big[k\,{}^3e\,{}^3R - \frac{1}{8k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}\,{}^3\tilde{\pi}^r_{(b)}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)}\Big](\tau,\vec{\sigma}) \approx 0,\\
\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) &= [\partial_r\,{}^3\tilde{\pi}^r_{(a)} - \epsilon_{(a)(b)(c)}\,{}^3\omega_{r(b)}\,{}^3\tilde{\pi}^r_{(c)}](\tau,\vec{\sigma}) = {}^3\tilde{\pi}^r_{(a)|r}(\tau,\vec{\sigma}) \approx 0,\\
&\Rightarrow \hat{H}_{(c)} = \int d^3\sigma[N\,\hat{\mathcal{H}} - N_{(a)}\,\hat{\mathcal{H}}_{(a)}](\tau,\vec{\sigma}) \approx 0.
\end{aligned}
\tag{61}
$$

It can be checked that the superhamiltonian constraint $\hat{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$ coincides with the ADM metric superhamiltonian one $\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$ given in Eqs.(79) of Section V, where also the ADM metric supermomentum constraints will be expressed in terms of the tetrad gravity constraints.

It is convenient to replace the constraints $\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma})\approx 0$ [they are of the type of SO(3) Yang-Mills Gauss laws, because they are the covariant divergence of a vector density] with the 3 constraints generating space pseudodiffeomorphisms on the cotriads and their conjugate momenta

$$\begin{aligned}
{}^3\tilde{\Theta}_r(\tau,\vec{\sigma}) &= -[{}^3e_{(a)r}\,\hat{\mathcal{H}}_{(a)} + {}^3\omega_{r(a)}\,{}^3\tilde{M}_{(a)}](\tau,\vec{\sigma}) = \\
&= [{}^3\tilde{\pi}^s_{(a)}\,\partial_r\,{}^3e_{(a)s} - \partial_s({}^3e_{(a)r}\,{}^3\tilde{\pi}^s_{(a)})](\tau,\vec{\sigma}) \approx 0,\\
\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) &= -{}^3e^r_{(a)}(\tau,\vec{\sigma})[{}^3\tilde{\Theta}_r + {}^3\omega_{r(b)}\,{}^3\tilde{M}_{(b)}](\tau,\vec{\sigma}) \approx 0,\\
&\Rightarrow \hat{H}^{'}_{(D)} = \hat{H}^{'}_{(c)} + \int d^3\sigma[\lambda_N\tilde{\pi}^N + \lambda^{\vec{N}}_{(a)}\tilde{\pi}^{\vec{N}}_{(a)} + \lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)} + \hat{\mu}_{(a)}\,{}^3\tilde{M}_{(a)}](\tau,\vec{\sigma}),\\
&\quad\ \hat{H}^{'}_{(c)} = \int d^3\sigma[N\hat{\mathcal{H}} + N_{(a)}\,{}^3e^r_{(a)}\,{}^3\tilde{\Theta}_r](\tau,\vec{\sigma}),
\end{aligned}\tag{62}$$

where we replaced $[\mu_{(a)} - N_{(b)}\,{}^3e^r_{(b)}\,{}^3\omega_{r(a)}](\tau,\vec{\sigma})$ with the new Dirac multipliers $\hat{\mu}_{(a)}(\tau,\vec{\sigma})$.

All the constraints are first class because the only non-identically vanishing Poisson brackets are

$$\begin{aligned}
\{{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}^{'})\} &= \epsilon_{(a)(b)(c)}\,{}^3\tilde{M}_{(c)}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
\{{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_r(\tau,\vec{\sigma}^{'})\} &= {}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}^{'})\,\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},\\
\{{}^3\tilde{\Theta}_r(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}^{'})\} &= [{}^3\tilde{\Theta}_r(\tau,\vec{\sigma}^{'})\frac{\partial}{\partial\sigma^s} + {}^3\tilde{\Theta}_s(\tau,\vec{\sigma})\frac{\partial}{\partial\sigma^r}]\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
\{\hat{\mathcal{H}}(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_r(\tau,\vec{\sigma}^{'})\} &= \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},\\
\{\hat{\mathcal{H}}(\tau,\vec{\sigma}), \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\} &= [{}^3e^r_{(a)}(\tau,\vec{\sigma})\,\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}) +\\
&\quad + {}^3e^r_{(a)}(\tau,\vec{\sigma}^{'})\,\hat{\mathcal{H}}_{(a)}(\tau,\vec{\sigma}^{'})]\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r} =\\
&\quad = \{[{}^3e^r_{(a)}\,{}^3e^s_{(a)}\,[{}^3\tilde{\Theta}_s + {}^3\omega_{s(b)}\,{}^3\tilde{M}_{(b)}]](\tau,\vec{\sigma}) +\\
&\quad + [{}^3e^r_{(a)}\,{}^3e^s_{(a)}\,[{}^3\tilde{\Theta}_s + {}^3\omega_{s(b)}\,{}^3\tilde{M}_{(b)}]](\tau,\vec{\sigma}^{'})\}\,\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r}.
\end{aligned}\tag{63}$$

The Poisson brackets of the cotriads and of their conjugate momenta with the constraints are $[{}^3R = \epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,{}^3e^s_{(b)}\,{}^3\Omega_{rs(c)}]$

$$\begin{aligned}
\{{}^3e_{(a)r}(\tau,\vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}^{'})\} &= \epsilon_{(a)(b)(c)}\,{}^3e_{(c)r}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
\{{}^3e_{(a)r}(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}^{'})\} &= \frac{\partial\,{}^3e_{(a)r}(\tau,\vec{\sigma})}{\partial\sigma^s}\delta^3(\vec{\sigma},\vec{\sigma}^{'}) + {}^3e_{(a)s}(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},\\
\{{}^3e_{(a)r}(\tau,\vec{\sigma}), \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\} &= -\frac{\epsilon}{4k}\Big[\frac{1}{{}^3e}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(b)r}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)}\Big](\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
\{{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau,\vec{\sigma}^{'})\} &= \epsilon_{(a)(b)(c)}\,{}^3\tilde{\pi}^r_{(c)}(\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}),
\end{aligned}$$

$$
\begin{aligned}
&\{{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}), {}^3\tilde{\Theta}_s(\tau,\vec{\sigma}^{'})\} = -{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}^{'})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^{'s}} + \delta^r_s\frac{\partial}{\partial\sigma^{'u}}[{}^3\tilde{\pi}^u_{(a)}(\tau,\vec{\sigma}^{'})\delta^3(\vec{\sigma},\vec{\sigma}^{'})],\\
&\{{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}), \hat{\mathcal{H}}(\tau,\vec{\sigma}^{'})\} = \epsilon\Big[2k\,{}^3e\,({}^3R^{rs} - \frac{1}{2}\,{}^3g^{rs}\,{}^3R)\,{}^3e_{(a)s} +\\
&+\frac{1}{4k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3\tilde{\pi}^r_{(b)}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)} -\\
&-\frac{1}{8k\,{}^3e}{}^3e^r_{(a)}\,{}^3G_{o(b)(c)(d)(e)}\,{}^3e_{(b)u}\,{}^3\tilde{\pi}^u_{(c)}\,{}^3e_{(d)v}\,{}^3\tilde{\pi}^v_{(e)}\Big](\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'}) +\\
&+2k\,{}^3e(\tau,\vec{\sigma})\Big[{}^3\Gamma^w_{uv}({}^3e^u_{(a)}\,{}^3g^{rv} - {}^3e^r_{(a)}\,{}^3g^{uv})\Big](\tau,\vec{\sigma}^{'})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^w} +\\
&+2k\,{}^3e(\tau,\vec{\sigma})\Big[{}^3e^u_{(a)}\,{}^3g^{rv} - {}^3e^r_{(a)}\,{}^3g^{uv})\Big](\tau,\vec{\sigma}^{'})\frac{\partial^2\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^u\partial\sigma^v},
\end{aligned}
\tag{64}
$$

where we used
$\{{}^3e(\tau,\vec{\sigma}){}^3R(\tau,\vec{\sigma}), {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}^{'})\} = -2k\Big[{}^3e({}^3R^{rs} - \frac{1}{2}{}^3g^{rs}\,{}^3R){}^3e_{(a)s}\Big](\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}^{'})+$
$+2k\,{}^3e(\tau,\vec{\sigma})\Big[{}^3\Gamma^w_{uv}({}^3e^u_{(a)}\,{}^3g^{rv} - {}^3e^r_{(a)}\,{}^3g^{uv})\Big](\tau,\vec{\sigma}^{'})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^w}+$
$+2k\,{}^3e(\tau,\vec{\sigma})\Big[{}^3e^u_{(a)}\,{}^3g^{rv} - {}^3e^r_{(a)}\,{}^3g^{uv}\Big](\tau,\vec{\sigma}^{'})\frac{\partial^2\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^u\partial\sigma^v}$.

The Hamilton equations associated with the Dirac Hamiltonian (62) are [see Eqs.(38) for ${}^3R^{uv}$]

$$
\begin{aligned}
\partial_\tau N(\tau,\vec{\sigma}) &\circeq \{N(\tau,\vec{\sigma}), \hat{H}^{'}_{(D)}\} = \lambda_N(\tau,\vec{\sigma}),\\
\partial_\tau N_{(a)}(\tau,\vec{\sigma}) &\circeq \{N_{(a)}(\tau,\vec{\sigma}), \hat{H}^{'}_{(D)}\} = \lambda^{\vec{N}}_{(a)}(\tau,\vec{\sigma}),\\
\partial_\tau \varphi_{(a)}(\tau,\vec{\sigma}) &\circeq \{\varphi_{(a)}(\tau,\vec{\sigma}), \hat{H}^{'}_{(D)}\} = \lambda^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}),\\
\partial_\tau\,{}^3e_{(a)r}(\tau,\vec{\sigma}) &\circeq \{{}^3e_{(a)r}(\tau,\vec{\sigma}), \hat{H}^{'}_{(D)}\} =\\
&= -\frac{\epsilon}{4k}\Big[\frac{N}{{}^3e}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(b)r}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)}\Big](\tau,\vec{\sigma}) +\\
&+\Big[N_{(b)}\,{}^3e^s_{(b)}\frac{\partial\,{}^3e_{(a)r}}{\partial\sigma^s} + {}^3e_{(a)s}\frac{\partial}{\partial\sigma^r}(N_{(b)}\,{}^3e^s_{(b)})\Big](\tau,\vec{\sigma}) +\\
&+\epsilon_{(a)(b)(c)}\,\hat{\mu}_{(b)}(\tau,\vec{\sigma})\,{}^3e_{(c)r}(\tau,\vec{\sigma}),\\
\partial_\tau\,{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &\circeq \{{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}), \hat{H}^{'}_{(D)}\} =\\
&= 2k\epsilon\Big[{}^3eN({}^3R^{rs} - \frac{1}{2}{}^3g^{rs}\,{}^3R){}^3e_{(a)s} + {}^3e(N^{|r|s} - {}^3g^{rs}\,N^{|u}{}_{|u}){}^3e_{(a)s}\Big](\tau,\vec{\sigma}) -\\
&-\epsilon\frac{N(\tau,\vec{\sigma})}{8k}\Big[\frac{1}{{}^3e}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3\tilde{\pi}^r_{(b)}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)} -\\
&-\frac{2}{{}^3e}\,{}^3e^r_{(a)}\,{}^3G_{o(b)(c)(d)(e)}\,{}^3e_{(b)u}\,{}^3\tilde{\pi}^u_{(c)}\,{}^3e_{(d)v}\,{}^3\tilde{\pi}^v_{(e)}\Big](\tau,\vec{\sigma}) +\\
&+\frac{\partial}{\partial\sigma^s}\Big[N_{(b)}\,{}^3e^s_{(b)}\,{}^3\tilde{\pi}^r_{(a)}\Big](\tau,\vec{\sigma}) - {}^3\tilde{\pi}^u_{(a)}(\tau,\vec{\sigma})\frac{\partial}{\partial\sigma^u}\Big[N_{(b)}\,{}^3e^r_{(b)}\Big](\tau,\vec{\sigma}) +\\
&+\epsilon_{(a)(b)(c)}\,\hat{\mu}_{(b)}(\tau,\vec{\sigma})\,{}^3\tilde{\pi}^r_{(c)}(\tau,\vec{\sigma}),\\
&\quad\Downarrow\\
\partial_\tau\,{}^3e^r_{(a)}(\tau,\vec{\sigma}) &= -\Big[{}^3e^r_{(b)}\,{}^3e^s_{(a)}\partial_\tau\,{}^3e_{(b)s}\Big](\tau,\vec{\sigma}) \circeq\\
&\circeq \frac{\epsilon}{4k}\Big[\frac{N}{{}^3e}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3e^r_{(b)}\,{}^3e_{(c)s}\,{}^3\tilde{\pi}^s_{(d)}\Big](\tau,\vec{\sigma}) -
\end{aligned}
$$

$$
\begin{aligned}
&- {}^3e^s_{(a)}\Big[N_{(c)}\, {}^3e^u_{(c)}\, {}^3e^r_{(b)} \frac{\partial\, {}^3e_{(b)s}}{\partial \sigma^u} + \frac{\partial}{\partial \sigma^s}(N_{(c)}\, {}^3e^r_{(c)})\Big](\tau,\vec{\sigma}) + \\
&+ \epsilon_{(a)(b)(c)}\, \hat{\mu}_{(b)}(\tau,\vec{\sigma})\, {}^3e^r_{(c)}(\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
\partial_\tau\, {}^3e(\tau,\vec{\sigma}) &= \Big[{}^3e\, {}^3e^r_{(a)} \partial_\tau\, {}^3e_{(a)r}\Big](\tau,\vec{\sigma}) \,{\buildrel \circ \over =}\\
&{\buildrel \circ \over =}\, \frac{\epsilon}{4k}\Big[N\, {}^3e_{(a)s}\, {}^3\tilde{\pi}^s_{(a)}\Big](\tau,\vec{\sigma}) + \\
&+ \Big({}^3e\,\Big[N_{(b)}\, {}^3e^s_{(b)}\, {}^3e^r_{(a)} \partial_s\, {}^3e_{(a)r} + {}^3e^r_{(a)}\, {}^3e_{(a)s} \partial_r (N_{(b)}\, {}^3e^s_{(b)})\Big]\Big)(\tau,\vec{\sigma}). \qquad (65)
\end{aligned}
$$

From the Hamilton equations and Eqs.(38), (57), we get

$$
\begin{aligned}
\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}) &{\buildrel \circ \over =}\, \Big[N_{r|s} + N_{s|r} - 2N\, {}^3K_{rs}\Big](\tau,\vec{\sigma}),\\
\partial_\tau\, {}^3K_{rs}(\tau,\vec{\sigma}) &{\buildrel \circ \over =}\, \frac{1}{4k}\, {}^3G_{o(a)(b)(c)(d)}\Big(\frac{\epsilon}{{}^3e}\Big[\partial_v (N_{(m)}\, {}^3e^v_{(m)}\, {}^3e_{(a)r}\, {}^3e_{(b)s}\, {}^3e_{(c)u}\, {}^3\tilde{\pi}^u_{(d)}) + \\
&+ {}^3e_{(c)u}\, {}^3\tilde{\pi}^u_{(d)}\Big[2kN({}^3R^{uv} - \frac{1}{2}\, {}^3g^{uv}\, {}^3R) + \epsilon(N^{|u|v} - {}^3g^{uv} N^{|l}{}_{|l})\Big]{}^3e_{(d)v} - \\
&- \frac{N}{4k\, {}^3e^2}\Big[\frac{1}{2}\, {}^3e_{(a)r}\, {}^3e_{(b)s}\, {}^3e_{(c)u}\, {}^3G_{o(d)(e)(f)(g)}\, {}^3\tilde{\pi}^u_{(e)}\, {}^3e_{(f)v}\, {}^3\tilde{\pi}^v_{(g)} - \\
&- {}^3e_{(a)r}\, {}^3e_{(b)s} \delta_{(c)(d)}\, {}^3G_{o(e)(f)(g)(h)}\, {}^3e_{(e)u}\, {}^3\tilde{\pi}^u_{(f)}\, {}^3e_{(g)v}\, {}^3\tilde{\pi}^v_{(h)} + \\
&+ {}^3e_{(a)r}\, {}^3e_{(b)s}({}^3e_{(m)v}\, {}^3\tilde{\pi}^v_{(m)}\, {}^3e_{(c)u}\, {}^3\tilde{\pi}^u_{(d)} + {}^3G_{o(c)(e)(f)(g)}\, {}^3e_{(e)u}\, {}^3\tilde{\pi}^u_{(d)}\, {}^3e_{(f)v}\, {}^3\tilde{\pi}^v_{(f)}) + \\
&+ ({}^3e_{(a)r}\, {}^3G_{o(b)(e)(f)(g)}\, {}^3e_{(e)s} + {}^3e_{(b)s}\, {}^3G_{o(a)(e)(f)(g)}\, {}^3e_{(e)r})\\
&\quad {}^3e_{(f)u}\, {}^3\tilde{\pi}^u_{(g)}\, {}^3e_{(c)v}\, {}^3\tilde{\pi}^v_{(d)}\Big]\Big)(\tau,\vec{\sigma}),\\
\partial_\tau\, {}^3K(\tau,\vec{\sigma}) &{\buildrel \circ \over =}\, \Big(\frac{1}{4}N\, {}^3R + 4N^{|r}{}_{|r} + \\
&+ \frac{N}{(4k\, {}^3e)^2}\Big[({}^3e_{(a)r}\, {}^3\tilde{\pi}^r_{(a)})^2 - \frac{3}{2}\, {}^3G_{o(a)(b)(c)(d)}\, {}^3e_{(a)r}\, {}^3\tilde{\pi}^r_{(b)}\, {}^3e_{(c)s}\, {}^3\tilde{\pi}^s_{(d)}\Big] - \\
&- \frac{\epsilon}{4k\, {}^3e}\, {}^3\tilde{\pi}^r_{(a)}\Big[N_{(m)}\, {}^3e^u_{(m)} \partial_u\, {}^3e_{(a)r} + {}^3e_{(a)u} \partial_r (N_{(m)}\, {}^3e^u_{(m)})\Big]\Big)(\tau,\vec{\sigma}),\\
\partial_\tau\, {}^3\omega_{r(a)(b)}(\tau,\vec{\sigma}) &{\buildrel \circ \over =}\, \frac{\epsilon N}{4k\, {}^3e}\Big((\partial_r\, {}^3e_{(b)s} - \partial_s\, {}^3e_{(b)r})^3G_{o(a)(l)(m)(n)}\, {}^3e^s_{(l)} + \\
&+ (\partial_s\, {}^3e_{(a)r} - \partial_r\, {}^3e_{(a)s})^3G_{o(b)(l)(m)(n)}\, {}^3e^s_{(l)} + \\
&+ (\partial_v\, {}^3e_{(c)u} - \partial_u\, {}^3e_{(c)v})\Big[{}^3e^v_{(b)}\, {}^3e_{(c)r}\, {}^3G_{o(a)(l)(m)(n)}\, {}^3e^u_{(l)} + \\
&+ {}^3e^u_{(a)}\, {}^3e_{(c)r}\, {}^3G_{o(b)(l)(m)(n)}\, {}^3e^v_{(l)} - \\
&- {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^3G_{o(c)(l)(m)(n)}\, {}^3e_{(l)r}\Big]{}^3e_{(m)t}\, {}^3\tilde{\pi}^t_{(n)}\Big) - \\
&- \frac{\epsilon}{4k}\Big(\Big[{}^3e^s_{(a)}\, {}^3G_{o(b)(l)(m)(n)} - {}^3e^s_{(b)}\, {}^3G_{o(a)(l)(m)(n)}\Big]\\
&\quad \Big[\partial_r(\frac{N}{{}^3e}\, {}^3e_{(l)s}\, {}^3e_{(m)t}\, {}^3\tilde{\pi}^t_{(n)}) - \partial_s(\frac{N}{{}^3e}\, {}^3e_{(l)r}\, {}^3e_{(m)t}\, {}^3\tilde{\pi}^t_{(n)})\Big] + \\
&+ {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^3e_{(c)r}\, {}^3G_{o(c)(l)(m)(n)}\\
&\quad \Big[\partial_v(\frac{N}{{}^3e}\, {}^3e_{(l)u}\, {}^3e_{(m)t}\, {}^3\tilde{\pi}^t_{(n)}) - \partial_u(\frac{N}{{}^3e}\, {}^3e_{(l)v}\, {}^3e_{(m)t}\, {}^3\tilde{\pi}^t_{(n)})\Big]\Big) - \\
&- \Big[(\partial_r\, {}^3e_{(b)s} - \partial_s\, {}^3e_{(b)r})^3e^v_{(a)} - (\partial_r\, {}^3e_{(a)s} - \partial_s\, {}^3e_{(a)r})^3e^v_{(b)}\Big]\\
&\quad \Big[N_{(w)}\, {}^3e^u_{(w)}\, {}^3e^s_{(l)} \partial_u (N_{(w)}\, {}^3e^s_{(w)})\Big] -
\end{aligned}
$$

$$
\begin{aligned}
&- (\partial_v\, {}^3e_{(c)u} - \partial_u\, {}^3e_{(c)v}) \Big[({}^3e^v_{(b)}\, {}^3e^t_{(a)} + {}^3e^u_{(a)}\, {}^3e^t_{(b)}) {}^3e_{(c)r} \\
&\quad \Big[N_{(m)}\, {}^3e^w_{(m)}\, {}^3e^u_{(l)} \partial_w\, {}^3e_{(l)t} + \partial_t (N_{(w)}\, {}^3e^u_{(w)}) \Big] + \\
&+ {}^3e^u_{(a)}\, {}^3e^v_{(b)} \Big(N_{(m)}\, {}^3e^w_{(m)} \partial_w\, {}^3e_{(c)r} + {}^3e_{(c)w} \partial_r (N_{(m)}\, {}^3e^w_{(m)}) \Big) \Big] + \\
&+ {}^3e^s_{(a)} \Big(\partial_r (N_{(w)}\, {}^3e^u_{(w)} \partial_u\, {}^3e_{(b)s} + {}^3e_{(b)u} \partial_s (N_{(w)}\, {}^3e^u_{(w)})) - \\
&- \partial_s (N_{(w)}\, {}^3e^u_{(w)} \partial_u\, {}^3e_{(b)r} + {}^3e_{(b)u} \partial_r (N_{(w)}\, {}^3e^u_{(w)})) \Big) - \\
&- {}^3e^s_{(b)} \Big(\partial_r (N_{(w)}\, {}^3e^u_{(w)} \partial_u\, {}^3e_{(a)s} + {}^3e_{(a)u} \partial_s (N_{(w)}\, {}^3e^u_{(w)})) - \\
&- \partial_s (N_{(w)}\, {}^3e^u_{(w)} \partial_u\, {}^3e_{(a)r} + {}^3e_{(a)u} \partial_r (N_{(w)}\, {}^3e^u_{(w)})) \Big) + \\
&+ {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^3e_{(c)r} \Big(\partial_v (N_{(w)}\, {}^3e^t_{(w)} \partial_t\, {}^3e_{(c)u} + {}^3e_{(c)t} \partial_u (N_{(w)}\, {}^3e^t_{(w)})) - \\
&- \partial_u (N_{(w)}\, {}^3e^t_{(w)} \partial_t\, {}^3e_{(c)v} + {}^3e_{(c)t} \partial_v (N_{(w)}\, {}^3e^t_{(w)})) \Big) + \\
&+ \Big(\Big[(\partial_r\, {}^3e_{(b)s} - \partial_s\, {}^3e_{(b)r}) \epsilon_{(a)(m)(n)} - (\partial_r\, {}^3e_{(a)s} - \partial_s\, {}^3e_{(a)r}) \epsilon_{(b)(m)(n)} \Big] {}^3e^s_{(n)} + \\
&+ (\partial_v\, {}^3e_{(c)u} - \partial_u\, {}^3e_{(c)v}) \Big[{}^3e^v_{(b)}\, {}^3e_{(c)r} \epsilon_{(a)(m)(n)}\, {}^3e^u_{(n)} + \\
&+ {}^3e^u_{(a)}\, {}^3e_{(c)r} \epsilon_{(b)(m)(n)}\, {}^3e^v_{(n)} + {}^3e^u_{(a)}\, {}^3e^v_{(b)} \epsilon_{(c)(m)(n)}\, {}^3e_{(n)r} \Big] \Big) \hat{\mu}_{(m)} + \\
&+ \Big[{}^3e^s_{(a)} \epsilon_{(b)(m)(n)} - {}^3e^s_{(b)} \epsilon_{(a)(m)(n)} \Big] \Big[\partial_r (\hat{\mu}_{(m)}\, {}^3e_{(n)s}) - \partial_s (\hat{\mu}_{(m)}\, {}^3e_{(n)r}) \Big] + \\
&+ {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^3e_{(c)r} \epsilon_{(c)(m)(n)} \Big[\partial_v (\hat{\mu}_{(m)}\, {}^3e_{(n)u}) - \partial_u (\hat{\mu}_{(m)}\, {}^3e_{(n)v}) \Big]. \qquad (66)
\end{aligned}
$$

They are needed in Appendix B, where there is the Hamiltonian version of the quantities given in Appendix A.

Let us consider the canonical transformation $\tilde{\pi}^N(\tau,\vec{\sigma})\, dN(\tau,\vec{\sigma}) + \tilde{\pi}^{\vec{N}}_{(a)}(\tau,\vec{\sigma})\, dN_{(a)}(\tau,\vec{\sigma}) + \tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma})\, d\varphi_{(a)}(\tau,\vec{\sigma}) + {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma})\, d^3e_{(a)r}(\tau,\vec{\sigma}) = {}^4\tilde{\pi}^A_{(\alpha)}(\tau,\vec{\sigma})\, d^4E^{(\alpha)}_A(\tau,\vec{\sigma})$, where ${}^4\tilde{\pi}^A_{(\alpha)}$ $[\{{}^4E^{(\alpha)}_A(\tau,\vec{\sigma}), {}^4\tilde{\pi}^B_{(\beta)}(\tau,\vec{\sigma}\,')\} = \delta^B_A \delta^{(\alpha)}_{(\beta)} \delta^3(\vec{\sigma},\vec{\sigma}\,')]$ would be the canonical momenta if the ADM action would be considered as a functional of the cotetrads ${}^4E^{(\alpha)}_A = {}^4E^{(\alpha)}_\mu\, b^\mu_A$ in the holonomic Σ_τ-adapted basis, as essentially is done in Refs. [68,67]. If $\bar{\gamma} = \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}$, we have

$$
\begin{aligned}
\tilde{\pi}^N &= (\bar{\gamma}\, {}^4\tilde{\pi}^\tau_{(o)} + \varphi^{(a)}\, {}^4\tilde{\pi}^\tau_{(a)}), \\
\tilde{\pi}^{\vec{N}}_{(a)} &= -\epsilon \varphi_{(a)}\, {}^4\tilde{\pi}^\tau_{(o)} + [\delta^{(b)}_{(a)} - \epsilon \frac{\varphi_{(a)} \varphi^{(b)}}{1+\bar{\gamma}}]\, {}^4\tilde{\pi}^\tau_{(b)}, \\
\tilde{\pi}^{\vec{\varphi}}_{(a)} &= (\frac{\epsilon N}{\bar{\gamma}}\, \varphi_{(a)} - N_{(a)})\, {}^4\tilde{\pi}^\tau_{(o)} - \delta^{(b)}_{(a)}\, N\, {}^4\tilde{\pi}^\tau_{(b)} - {}^3e_{(a)r}\, {}^4\tilde{\pi}^r_{(o)} - \\
&- \frac{1}{1+\bar{\gamma}} (\delta^{(c)}_{(a)} \varphi^{(b)} + \delta^{(b)}_{(a)} \varphi^{(c)} + \epsilon \frac{\varphi_{(a)} \varphi^{(b)} \varphi^{(c)}}{\bar{\gamma}(1+\bar{\gamma})}) (N_{(c)}\, {}^4\tilde{\pi}^\tau_{(b)} + {}^3e_{(c)r}\, {}^4\tilde{\pi}^r_{(b)}), \\
{}^3\tilde{\pi}^r_{(a)} &= -\epsilon \varphi_{(a)}\, {}^4\tilde{\pi}^r_{(o)} + (\delta^{(b)}_{(a)} - \epsilon \frac{\varphi_{(a)} \varphi^{(b)}}{1+\bar{\gamma}})\, {}^4\tilde{\pi}^r_{(b)}, \\
{}^4\tilde{\pi}^\tau_{(o)} &= \bar{\gamma} \tilde{\pi}^N - \varphi^{(a)}\, {}^3\tilde{\pi}^{\vec{N}}_{(a)}, \\
{}^4\tilde{\pi}^\tau_{(a)} &= \epsilon \varphi_{(a)} \tilde{\pi}^N + [\delta^{(b)}_{(a)} - \epsilon \frac{\varphi_{(a)} \varphi^{(b)}}{1+\bar{\gamma}}] \tilde{\pi}^{\vec{N}}_{(b)},
\end{aligned}
$$

$$
\begin{aligned}
{}^4\tilde{\pi}^r_{(o)} &= -\bar{\gamma}\, {}^3e^r_{(a)} [\delta^{(b)}_{(a)} - \epsilon \frac{\varphi_{(a)}\varphi^{(b)}}{1+\bar{\gamma}}] \tilde{\pi}^{\vec{\varphi}}_{(b)} + \bar{\gamma} N_{(a)}\, {}^3e^r_{(a)}\, \tilde{\pi}^N + \\
&+ {}^3e^r_{(a)} [-N\, \delta^{(b)}_{(a)} - (\delta^{(b)}_{(a)}\varphi^{(c)} - \delta^{(c)}_{(a)}\varphi^{(b)}) \frac{N_{(c)}}{1+\bar{\gamma}}] \tilde{\pi}^{\vec{N}}_{(b)} - \\
&- \frac{1}{1+\bar{\gamma}}\, {}^3e^r_{(a)} [\delta^{(a)(b)}\varphi^{(c)} + \frac{1}{\bar{\gamma}} \delta^{(b)(c)}\varphi^{(a)}]^3 e_{(c)s}\, {}^3\tilde{\pi}^s_{(b)}, \\
{}^4\tilde{\pi}^r_{(a)} &= [\delta^{(b)}_{(a)} + \epsilon \frac{\varphi_{(a)}\varphi^{(b)}}{\bar{\gamma}(1+\bar{\gamma})}]^3 \tilde{\pi}^r_{(b)} - \epsilon\varphi_{(a)}\, {}^3e^r_{(b)} [\delta^{(c)}_{(b)} - \epsilon \frac{\varphi_{(b)}\varphi^{(c)}}{1+\bar{\gamma}}] \tilde{\pi}^{\vec{\varphi}}_{(c)} + \\
&+ \epsilon\varphi_{(a)}\, {}^3e^r_{(b)} N_{(b)} \tilde{\pi}^N - \epsilon\varphi_{(a)}\, {}^3e^r_{(b)} [N \delta^{(b)}_{(c)} + (\delta^{(b)}_{(c)}\varphi^{(d)} - \delta^{(d)}_{(c)}\varphi^{(b)}) \frac{N_{(d)}}{1+\bar{\gamma}}] \tilde{\pi}^{\vec{N}}_{(c)} - \\
&- \epsilon \frac{\varphi_{(a)}}{1+\bar{\gamma}} [\varphi^{(b)}\delta^{(c)}_{(d)} + \frac{1}{\bar{\gamma}}\varphi^{(c)}\delta^{(b)}_{(d)}]^3 e^r_{(c)}\, {}^3e_{(b)s}\, {}^3\tilde{\pi}^s_{(d)}.
\end{aligned}
\tag{67}
$$

Our canonical transformation (67) allows to consider the metric ADM Lagrangian as function of the cotetrads ${}^4E^{(\alpha)}_A = {}^4E^{(\alpha)}_\mu\, b^\mu_A$ and to find the conjugate momenta ${}^4\tilde{\pi}^A_{(\alpha)}$. Eqs.(67) show that the four primary constraints, which contain the informations $\tilde{\pi}^N \approx 0$ and $\tilde{\pi}^{\vec{N}}_{(a)} \approx 0$, are ${}^4\tilde{\pi}^\tau_{(\alpha)} \approx 0$. The six primary constraints (the generators of the local Lorentz transformations) ${}^4\tilde{M}_{(\alpha)(\beta)} = {}^4E^{(\gamma)}_A [{}^4\eta_{(\alpha)(\gamma)}\, {}^4\tilde{\pi}^A_{(\beta)} - {}^4\eta_{(\beta)(\gamma)}\, {}^4\tilde{\pi}^A_{(\alpha)}] \approx 0$ of this formulation have the following relation with $\tilde{\pi}^{\vec{\varphi}}_{(a)} \approx 0$ and ${}^3\tilde{M}_{(a)} \approx 0$

$$
\begin{aligned}
{}^4\tilde{M}_{(a)(b)} &= -\epsilon^3 \tilde{M}_{(a)(b)} + (\varphi_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(b)} - \varphi_{(b)}\tilde{\pi}^{\vec{\varphi}}_{(a)}) + \epsilon(\varphi_{(a)}N_{(b)} - \varphi_{(b)}N_{(a)})\tilde{\pi}^N - \\
&- (\delta^{(c)}_{(a)}\delta^{(d)}_{(b)} - \delta^{(c)}_{(b)}\delta^{(d)}_{(a)})[\epsilon N \varphi_{(c)}\delta_{(d)(e)} + \\
&+ (\delta_{(c)(f)} + \frac{\varphi_{(c)}\varphi_{(f)}}{1+\bar{\gamma}})(\delta_{(d)(e)} + \frac{\varphi_{(d)}\varphi_{(e)}}{1+\bar{\gamma}}) N_{(f)}] \tilde{\pi}^{\vec{N}}_{(e)} \approx 0, \\
{}^4\tilde{M}_{(a)(o)} &= -\epsilon\bar{\gamma}\tilde{\pi}^{\vec{\varphi}}_{(a)} - \frac{1}{1+\bar{\gamma}}\, {}^3\tilde{M}_{(a)(b)}\varphi_{(b)} - \epsilon\bar{\gamma}(\delta_{(a)(b)} - \frac{\varphi_{(a)}\varphi_{(b)}}{\bar{\gamma}(1+\bar{\gamma})}) N_{(b)}\tilde{\pi}^N + \\
&+ [-\epsilon\bar{\gamma} N(\delta_{(a)(b)} - \frac{\varphi_{(a)}\varphi_{(b)}}{\bar{\gamma}(1+\bar{\gamma})}) + \varphi_{(c)}N_{(c)}\delta_{(a)(b)} - N_{(a)}\varphi_{(b)}] \tilde{\pi}^{\vec{N}}_{(b)} \approx 0, \\
{}^3\tilde{M}_{(a)(b)} &= -\epsilon^4 \tilde{M}_{(a)(b)} + \frac{\epsilon}{1+\bar{\gamma}} [\varphi_{(a)}\, {}^4\tilde{M}_{(b)(c)} - \varphi_{(b)}\, {}^4\tilde{M}_{(a)(c)}] \varphi_{(c)} + \\
&+ [\varphi_{(a)}\, {}^4\tilde{M}_{(b)(o)} - \varphi_{(b)}\, {}^4\tilde{M}_{(a)(o)}] - [\varphi_{(a)}\, {}^4E^\tau_{(b)} - \varphi_{(b)}\, {}^4E^\tau_{(a)}]^4 \tilde{\pi}^\tau_{(o)} - \\
&- \epsilon[(\delta_{(a)(c)}\delta_{(d)(e)} - \delta_{(a)(e)}\delta_{(b)(c)})(\delta_{(c)(d)} + \frac{\varphi_{(c)}\varphi_{(d)}}{1+\bar{\gamma}})^4 E^\tau_{(c)} + \\
&+ \epsilon({}^4E^\tau_{(o)} + \epsilon \frac{\varphi_{(c)}\, {}^4E^\tau_{(c)}}{1+\bar{\gamma}})(\delta_{(a)(d)}\varphi_{(b)} - \delta_{(b)(d)}\varphi_{(a)})]^4 \tilde{\pi}^\tau_{(d)} \approx 0, \\
\tilde{\pi}^{\vec{\varphi}}_{(a)} &= \epsilon(\delta_{(a)(b)} - \frac{\varphi_{(a)}\varphi_{(b)}}{\bar{\gamma}(1+\bar{\gamma})})^4 \tilde{M}_{(b)(o)} + \frac{1}{1+\bar{\gamma}}\, {}^4\tilde{M}_{(a)(b)}\varphi_{(b)} - \\
&- \epsilon(\delta_{(a)(b)} - \frac{\varphi_{(a)}\varphi_{(b)}}{\bar{\gamma}(1+\bar{\gamma})})\, {}^4E^\tau_{(b)}\, {}^4\tilde{\pi}^\tau_{(o)} + \\
&+ [\epsilon(\delta_{(a)(b)} - \frac{\varphi_{(a)}\varphi_{(b)}}{\bar{\gamma}(1+\bar{\gamma})})\, {}^4E^\tau_{(o)} - \frac{\varphi_{(c)}}{1+\bar{\gamma}}(\delta_{(c)(b)}\, {}^4E^\tau_{(a)} - \delta_{(c)(a)}\, {}^4E^\tau_{(b)})]^4 \tilde{\pi}^\tau_{(b)} \approx 0.
\end{aligned}
\tag{68}
$$

Let us add a comment on the literature on tetrad gravity. The use of tetrads (or vierbeins or local frames) started with Ref. [58], where vierbeins and spin connections are used as independent variables in a Palatini form of the Lagrangian. They were used by Dirac [59] for the coupling of gravity to fermion fields (see also Ref. [66]) and here Σ_τ-adapted tetrads were introduced. In Ref. [60] the reduction of this theory at the Lagrangian level was done by introducing the so-called 'time-gauge' ${}^4E^{(o)}_r = 0$ [or ${}^4E^o_{(a)} = 0$], which distinguishes the time coordinate $x^o = const.$ planes; in this paper there is also the coupling to scalar fields, while in Ref. [61] the coupling to Dirac-Maiorana fields is studied. In Ref. [63] there is a non-metric Lagrangian formulation, see Eq.(29), employing as basic variables the cotetrads ${}^4E^{(\alpha)}_\mu$, which is different from our metric Lagrangian and has different primary constraints; its Hamiltonian formulation is completely developed. See also Ref. [64] for a study of the tetrad frame constraint algebra. In the fourth of Refs. [62] cotetrads ${}^4E^{(\alpha)}_\mu$ together with the spin connection ${}^4\omega^{(\alpha)}_{\mu(\beta)}$ are used as independent variables in a first order Palatini action [see also the Nelson-Regge papers in Refs. [62] for a different approach, the so-called covariant canonical formalism], while in Ref. [65] a first order Lagrangian reformulation is done for Eq.(29) [in both these papers there is a 3+1 decomposition of the tetrads different from our and, like in Ref. [65], use is done of the Schwinger time gauge to get free of three boost-like parameters].

Instead in most of Refs. [62,67,68] one uses the space components ${}^4E^{(\alpha)}_r$ of cotetrads ${}^4E^{(\alpha)}_\mu$, together with the conjugate momenta ${}^4\tilde{\pi}^r_{(\alpha)}$ inside the ADM Hamiltonian, in which one puts ${}^3g_{rs} = {}^4E^{(\alpha)}_r\, {}^4\eta_{(\alpha)(\beta)}\, {}^4E^{(\beta)}_s$ and ${}^3\tilde{\Pi}^{rs} = \frac{1}{4}{}^4\eta^{(\alpha)(\beta)}[{}^4E^r_{(\alpha)}\; {}^4\tilde{\pi}^s_{(\beta)} + {}^4E^s_{(\alpha)}\; {}^4\tilde{\pi}^r_{(\beta)}]$. Lapse and shift functions are treated as Hamiltonian multipliers and there is no worked out Lagrangian formulation. In Ref. [69] it is shown how to go from the space components ${}^4E^{(\alpha)}_r$ to cotriads ${}^3e_{(a)r}$ by using the "time gauge" on a surface $x^0 = const.$; here it is introduced for the first time the concept of parameters of Lorentz boosts [if they are put equal to zero, one recovers Schwinger's time gauge], which was our starting point to arrive at the identification of the Wigner boost parameters $\varphi_{(a)}$. Finally in Ref. [85] there is a 3+1 decomposition of tetrads and cotetrads in which some boost-like parameters have been fixed (it is a Schwinger time gauge) so that one can arrive at a Lagrangian (different from ours) depending only on lapse, shift and cotriads.

In Ref. [69] there is another canonical transformation from cotriads and their conjugate momenta to a new canonical basis containing densitized triads and their conjugate momenta

$$\begin{aligned}
(\,{}^3e_{(a)r}\;,\;\; {}^3\tilde{\pi}^r_{(a)}\,) \mapsto (\;\; & {}^3\tilde{h}^r_{(a)} = {}^3e\, {}^3e^r_{(a)},\\
& 2\, {}^3K_{(a)r} = 2[{}^3e^s_{(a)}\; {}^3K_{sr} + \frac{1}{4\, {}^3e}{}^3\tilde{M}_{(a)(b)}\; {}^3e_{(b)r}] =\\
= \frac{1}{2}[\frac{1}{k}{}^3G_{o(a)(b)(c)(d)}\; & {}^3e_{(b)r}\; {}^3e_{(c)u}\; {}^3\tilde{\pi}^u_{(d)} + \frac{1}{{}^3e}{}^3\tilde{M}_{(a)(b)}\; {}^3e_{(b)r}]\,),
\end{aligned} \tag{69}$$

which is used to make the transition to the complex Ashtekar variables [37]

$$(\,{}^3\tilde{h}^r_{(a)}, \qquad {}^3A_{(a)r} = 2\; {}^3K_{(a)r} + i\, {}^3\omega_{r(a)}\,), \tag{70}$$

where ${}^3A_{(a)r}$ is a zero density whose real part (in this notation) can be considered the gauge potential of the Sen connection and plays an important role in the simplification of the functional form of the constraints present in this approach; the conjugate variable is a density 1 SU(2) soldering form.

V. COMPARISON WITH ADM CANONICAL METRIC GRAVITY.

In this Section we give a brief review of the Hamiltonian formulation of ADM metric gravity [see Refs. [78,71,27,77,79]] to express its constraints in terms of those of Section IV.

Let us rewrite Eq.(25) in terms of the independent variables N, $N_r = {}^3g_{rs}N^s$, ${}^3g_{rs}$ as $S_{ADM} = \int d\tau\, L_{ADM}(\tau) = \int d\tau d^3\sigma \mathcal{L}_{ADM}(\tau,\vec{\sigma}) = -\epsilon k \int_{\triangle\tau} d\tau \int d^3\sigma \{\sqrt{\gamma} N\, [{}^3R + {}^3K_{rs}\, {}^3K^{rs} - ({}^3K)^2]\}(\tau,\vec{\sigma})$. Since $\delta_o(\sqrt{\gamma}{}^3R) = \sqrt{\gamma}({}^3R_{rs} - \frac{1}{2}\, {}^3g_{rs}\, {}^3R)\delta_o\, {}^3g^{rs} + \sqrt{\gamma}({}^3g_{rs}\delta_o\, {}^3g^{rs|u} - \delta_o\, {}^3g^{ru}{}_{|r})_{|u}$, we get

$\delta_o S_{ADM} = \int d\tau d^3\sigma \Big(L_N \delta_o N + L^r_{\vec{N}} \delta_o N_r + L^{rs}_g \delta_o\, {}^3g_{rs}$
$+\partial_\tau \Big[\epsilon k \sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\, {}^3K)\delta_o\, {}^3g_{rs}\Big] - \partial_r \Big[2\epsilon k \sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\, {}^3K)\delta_o\, N_s\Big]$
$+\partial_r \Big[\epsilon k \sqrt{\gamma}(N[{}^3g^{uv}\delta_o\, {}^3g_{uv}{}^{|r} - {}^3g^{ur}\delta_o\, {}^3g_{uv}{}^{|v}] + N^{|u}\, {}^3g^{rs}\delta_o\, {}^3g_{us} - N^{|r}\, {}^3g^{uv}\delta_o\, {}^3g_{uv}\Big]\Big)$,

so that the Euler-Lagrange equations are

$$\begin{aligned}
L_N &= \frac{\partial \mathcal{L}_{ADM}}{\partial N} - \partial_\tau \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_\tau N} - \partial_r \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_r N} = \\
&= -\epsilon k \sqrt{\gamma}[{}^3R - {}^3K_{rs}\, {}^3K^{rs} + ({}^3K)^2] = -2\epsilon k\, {}^4\bar{G}_{ll} \stackrel{\circ}{=} 0, \\
L^r_{\vec{N}} &= \frac{\partial \mathcal{L}_{ADM}}{\partial N_r} - \partial_\tau \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_\tau N_r} - \partial_s \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_s N_r} = \\
&= 2\epsilon k[\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\, {}^3K)]_{|s} = 2k\, {}^4\bar{G}_l{}^r \stackrel{\circ}{=} 0, \\
L^{rs}_g &= -\epsilon k\Big[\frac{\partial}{\partial \tau}[\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\, {}^3K)] - N\sqrt{\gamma}({}^3R^{rs} - \frac{1}{2}{}^3g^{rs}\, {}^3R) + \\
&\quad +2N\sqrt{\gamma}({}^3K^{ru}\, {}^3K_u{}^s - {}^3K\, {}^3K^{rs}) + \frac{1}{2}N\sqrt{\gamma}[({}^3K)^2 - {}^3K_{uv}\, {}^3K^{uv})\, {}^3g^{rs} + \\
&\quad +\sqrt{\gamma}({}^3g^{rs}N^{|u}{}_{|u} - N^{|r|s})\Big] = -\epsilon k N \sqrt{\gamma}\, {}^4\bar{G}^{rs} \stackrel{\circ}{=} 0,
\end{aligned} \tag{71}$$

and correspond to the Einstein equations in the form ${}^4\bar{G}_{ll} \stackrel{\circ}{=} 0$, ${}^4\bar{G}_{lr} \stackrel{\circ}{=} 0$, ${}^4\bar{G}_{rs} \stackrel{\circ}{=} 0$, respectively. As shown after Eq.(10) there are four contracted Bianchi identities implying that only two of the equations $L^{rs}_g \stackrel{\circ}{=} 0$ are independent.

The canonical momenta (densities of weight -1) are

$$\begin{aligned}
\tilde{\Pi}^N(\tau,\vec{\sigma}) &= \frac{\delta S_{ADM}}{\delta \partial_\tau N(\tau,\vec{\sigma})} = 0, \\
\tilde{\Pi}^r_{\vec{N}}(\tau,\vec{\sigma}) &= \frac{\delta S_{ADM}}{\delta \partial_\tau N_r(\tau,\vec{\sigma})} = 0, \\
{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) &= \frac{\delta S_{ADM}}{\delta \partial_\tau {}^3g_{rs}(\tau,\vec{\sigma})} = \epsilon k\, [\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\, {}^3K)](\tau,\vec{\sigma}), \\
{}^3K_{rs} &= \frac{\epsilon}{k\sqrt{\gamma}}[{}^3\tilde{\Pi}_{rs} - \frac{1}{2}{}^3g_{rs}\, {}^3\tilde{\Pi}], \qquad {}^3\tilde{\Pi} = {}^3g_{rs}\, {}^3\tilde{\Pi}^{rs} = -2\epsilon k \sqrt{\gamma}\, {}^3K,
\end{aligned} \tag{72}$$

and satisfy the Poisson brackets

$$\begin{aligned}
&\{N(\tau,\vec{\sigma}), \tilde{\Pi}^N(\tau,\vec{\sigma}^{'})\} = \delta^3(\vec{\sigma},\vec{\sigma}^{'}), \\
&\{N_r(\tau,\vec{\sigma}), \tilde{\Pi}^s_{\vec{N}}(\tau,\vec{\sigma}^{'})\} = \delta^s_r \delta^3(\vec{\sigma},\vec{\sigma}^{'}), \\
&\{{}^3g_{rs}(\tau,\vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma}^{'}\} = \frac{1}{2}(\delta^u_r\delta^v_s + \delta^v_r\delta^u_s)\delta^3(\vec{\sigma},\vec{\sigma}^{'}).
\end{aligned} \tag{73}$$

Let us introduce a new tensor, the Wheeler- DeWitt supermetric

$$ {}^3G_{rstw}(\tau,\vec{\sigma}) = [{}^3g_{rt}\,{}^3g_{sw} + {}^3g_{rw}\,{}^3g_{st} - {}^3g_{rs}\,{}^3g_{tw}](\tau,\vec{\sigma}), \tag{74} $$

whose inverse is defined by the equations

$$ \begin{aligned} \frac{1}{2}{}^3G_{rstw}\,\frac{1}{2}{}^3G^{twuv} &= \frac{1}{2}(\delta^u_r\delta^v_s + \delta^v_r\delta^u_s),\\ {}^3G^{twuv}(\tau,\vec{\sigma}) &= [{}^3g^{tu}\,{}^3g^{wv} + {}^3g^{tv}\,{}^3g^{wu} - 2\,{}^3g^{tw}\,{}^3g^{uv}](\tau,\vec{\sigma}), \end{aligned} \tag{75} $$

so that we get

$$ \begin{aligned} {}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) &= \frac{1}{2}\epsilon k\sqrt{\gamma}\,{}^3G^{rsuv}(\tau,\vec{\sigma})\,{}^3K_{uv}(\tau,\vec{\sigma}),\\ {}^3K_{rs}(\tau,\vec{\sigma}) &= \frac{\epsilon}{2k\sqrt{\gamma}}\,{}^3G_{rsuv}(\tau,\vec{\sigma})\,{}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma}),\\ &\quad [{}^3K^{rs}\,{}^3K_{rs} - ({}^3K)^2](\tau,\vec{\sigma}) =\\ &\quad = k^{-2}[\gamma^{-1}({}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}_{rs} - \frac{1}{2}({}^3\tilde{\Pi})^2](\tau,\vec{\sigma}) = (2k)^{-1}[\gamma^{-1}\,{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}^{uv}](\tau,\vec{\sigma}),\\ \partial_\tau\,{}^3g_{rs}(\tau,\vec{\sigma}) &= [N_{r|s} + N_{s|r} - \frac{\epsilon N}{k\sqrt{\gamma}}\,{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{uv}](\tau,\vec{\sigma}). \end{aligned} \tag{76} $$

Since ${}^3\tilde{\Pi}^{rs}\partial_\tau\,{}^3g_{rs} = {}^3\tilde{\Pi}^{rs}[N_{r|s}+N_{s|r}-\frac{\epsilon N}{k\sqrt{\gamma}}{}^3G_{rsuv}{}^3\tilde{\Pi}^{uv}] = -2N_r\,{}^3\tilde{\Pi}^{rs}{}_{|s} - \frac{\epsilon N}{k\sqrt{\gamma}}\,{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{rs}{}^3\tilde{\Pi}^{uv} + (2N_r\,{}^3\tilde{\Pi}^{rs})_{|s}$, we obtain the canonical Hamiltonian [since $N_r\,{}^3\tilde{\Pi}^{rs}$ is a vector density of weight -1, we have ${}^3\nabla_s(N_r\,{}^3\tilde{\Pi}^{rs}) = \partial_s(N_r\,{}^3\tilde{\Pi}^{rs})$]

$$ \begin{aligned} H_{(c)ADM} &= \int_S d^3\sigma\,[\tilde{\Pi}^N\partial_\tau N + \tilde{\Pi}^r_{\vec{N}}\partial_\tau N_r + {}^3\tilde{\Pi}^{rs}\partial_\tau{}^3g_{rs}](\tau,\vec{\sigma}) - L_{ADM} =\\ &= \int_S d^3\sigma\,[\epsilon N(k\sqrt{\gamma}\,{}^3R - \frac{1}{2k\sqrt{\gamma}}{}^3G_{rsuv}{}^3\tilde{\Pi}^{rs}{}^3\tilde{\Pi}^{uv}) - 2N_r\,{}^3\tilde{\Pi}^{rs}{}_{|s}](\tau,\vec{\sigma}) +\\ &+ 2\int_{\partial S} d^2\Sigma_s[N_r\,{}^3\tilde{\Pi}^{rs}\,](\tau,\vec{\sigma}), \end{aligned} \tag{77} $$

In the following discussion we shall omit the surface term.

The Dirac Hamiltonian is [the $\lambda(\tau,\vec{\sigma})$'s are arbitrary Dirac multipliers]

$$ H_{(D)ADM} = H_{(c)ADM} + \int d^3\sigma\,[\lambda_N\,\tilde{\Pi}^N + \lambda^{\vec{N}}_r\,\tilde{\Pi}^r_{\vec{N}}](\tau,\vec{\sigma}). \tag{78} $$

The τ-constancy of the primary constraints [$\partial_\tau\tilde{\Pi}^N(\tau,\vec{\sigma}) = \{\tilde{\Pi}^N(\tau,\vec{\sigma}), H_{(D)ADM}\} \approx 0$, $\partial_\tau\tilde{\Pi}^r_{\vec{N}}(\tau,\vec{\sigma}) = \{\tilde{\Pi}^r_{\vec{N}}(\tau,\vec{\sigma}), H_{(D)ADM}\} \approx 0$] generates four secondary constraints [all 4 are densities of weight -1] which correspond to the Einstein equations ${}^4\bar{G}_{ll}(\tau,\vec{\sigma}) \circeq 0$, ${}^4\bar{G}_{lr}(\tau,\vec{\sigma}) \circeq 0$ [see after Eqs.(10)]

$$ \begin{aligned} \tilde{\mathcal{H}}(\tau,\vec{\sigma}) &= \epsilon[k\sqrt{\gamma}\,{}^3R - \frac{1}{2k\sqrt{\gamma}}{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}^{uv}](\tau,\vec{\sigma}) =\\ &= \epsilon[\sqrt{\gamma}\,{}^3R - \frac{1}{k\sqrt{\gamma}}({}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}_{rs} - \frac{1}{2}({}^3\tilde{\Pi})^2)](\tau,\vec{\sigma}) =\\ &= \epsilon k\{\sqrt{\gamma}[{}^3R - ({}^3K_{rs}\,{}^3K^{rs} - ({}^3K)^2)]\}(\tau,\vec{\sigma}) \approx 0,\\ {}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) &= -2\,{}^3\tilde{\Pi}^{rs}{}_{|s}(\tau,\vec{\sigma}) = -2[\partial_s\,{}^3\tilde{\Pi}^{rs} + {}^3\Gamma^r_{su}\,{}^3\tilde{\Pi}^{su}](\tau,\vec{\sigma}) =\\ &= -2\epsilon k\{\partial_s[\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\,{}^3K)] + {}^3\Gamma^r_{su}\sqrt{\gamma}({}^3K^{su} - {}^3g^{su}\,{}^3K)\}(\tau,\vec{\sigma}) \approx 0, \end{aligned} \tag{79} $$

so that we have

$$H_{(c)ADM} = \int d^3\sigma [N\,\tilde{\mathcal{H}} + N_r\,{}^3\tilde{\mathcal{H}}^r](\tau,\vec{\sigma}) \approx 0, \tag{80}$$

with $\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$ called the superhamiltonian constraint and ${}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) \approx 0$ called the supermomentum constraints. See Ref. [86] for their interpretation as the generators of the change of the canonical data ${}^3g_{rs}$, ${}^3\tilde{\Pi}^{rs}$, under the normal and tangent deformations of the spacelike hypersurface Σ_τ which generate $\Sigma_{\tau+d\tau}$ [one thinks to Σ_τ as determined by a cloud of observers, one per space point; the idea of bifurcation and reencounter of the observers is expressed by saying that the data on Σ_τ (where the bifurcation took place) are propagated to some final $\Sigma_{\tau+d\tau}$ (where the reencounter arises) along different intermediate paths, each path being a monoparametric family of surfaces that fills the sandwich in between the two surfaces; embeddability of Σ_τ in M^4 becomes the synonymous with path independence; see also Ref. [82] for the connection with the theorema egregium of Gauss).

In $\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$ one can say that the term $-\epsilon k\sqrt{\gamma}({}^3K_{rs}\,{}^3K^{rs} - {}^3K^2)$ is the kinetic energy and $\epsilon k\sqrt{\gamma}\,{}^3R$ the potential energy: in any Ricci flat spacetime (i.e. one satisfying Einstein's empty-space equations) the extrinsic and intrinsic scalar curvatures of any spacelike hypersurface Σ_τ are both equal to zero (also the converse is true [87]).

All the constraints are first class, because the only non-identically zero Poisson brackets correspond to the so called universal Dirac algebra [1]:

$$\begin{aligned}
\{{}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}), {}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}^{'})\} &= \\
&= {}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}^{'})\,\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^s} + {}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},\\
\{\tilde{\mathcal{H}}(\tau,\vec{\sigma}), {}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}^{'})\} &= \tilde{\mathcal{H}}(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},\\
\{\tilde{\mathcal{H}}(\tau,\vec{\sigma}), \tilde{\mathcal{H}}(\tau,\vec{\sigma}^{'})\} &= [{}^3g^{rs}(\tau,\vec{\sigma})^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}) + \\
&+ {}^3g^{rs}(\tau,\vec{\sigma}^{'})^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}^{'})]\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},
\end{aligned} \tag{81}$$

with ${}^3\tilde{\mathcal{H}}_r = {}^3g_{rs}\,{}^3\tilde{\mathcal{H}}^r$ as the combination of the supermomentum constraints satisfying the algebra of 3-diffeomorphisms. In Ref. [86] it is shown that Eqs.(81) are sufficient conditions for the embeddability of Σ_τ into M^4. In the second paper in Ref. [9] it is shown that the last two lines of the Dirac algebra are the equivalent in phase space of the Bianchi identities ${}^4G^{\mu\nu}{}_{;\nu} \equiv 0$.

The Hamilton-Dirac equations are

$$\begin{aligned}
\partial_\tau N(\tau,\vec{\sigma}) &\,\circeq\, \{N(\tau,\vec{\sigma}), H_{(D)ADM}\} = \lambda_N(\tau,\vec{\sigma}),\\
\partial_\tau N_r(\tau,\vec{\sigma}) &\,\circeq\, \{N_r(\tau,\vec{\sigma}), H_{(D)ADM}\} = \lambda^{\vec{N}}_r(\tau,\vec{\sigma}),\\
\partial_\tau\,{}^3g_{rs}(\tau,\vec{\sigma}) &\,\circeq\, \{{}^3g_{rs}(\tau,\vec{\sigma}), H_{(D)ADM}\} = [N_{r|s} + N_{s|r} - \frac{2\epsilon N}{k\sqrt{\gamma}}({}^3\tilde{\Pi}_{rs} - \frac{1}{2}{}^3g_{rs}\,{}^3\tilde{\Pi})](\tau,\vec{\sigma}) =\\
&= [N_{r|s} + N_{s|r} - 2N\,{}^3K_{rs}](\tau,\vec{\sigma}),\\
\partial_\tau\,{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) &\,\circeq\, \{{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}), H_{(D)ADM}\} = \epsilon[N\,k\sqrt{\gamma}({}^3R^{rs} - \frac{1}{2}{}^3g^{rs}\,{}^3R)](\tau,\vec{\sigma}) -
\end{aligned}$$

$$
\begin{aligned}
&- 2\epsilon[\frac{N}{k\sqrt{\gamma}}(\frac{1}{2}{}^3\tilde{\Pi}\,{}^3\tilde{\Pi}^{rs} - {}^3\tilde{\Pi}^r{}_u\,{}^3\tilde{\Pi}^{us})(\tau,\vec{\sigma}) -\\
&- \frac{\epsilon N}{2}\frac{{}^3g^{rs}}{k\sqrt{\gamma}}(\frac{1}{2}{}^3\tilde{\Pi}^2 - {}^3\tilde{\Pi}_{uv}\,{}^3\tilde{\Pi}^{uv})](\tau,\vec{\sigma}) +\\
&+ \mathcal{L}_{\vec{N}}\,{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) + \epsilon[k\sqrt{\gamma}(N^{|r|s} - {}^3g^{rs}\,N^{|u}{}_{|u})](\tau,\vec{\sigma}),\\
&with \qquad \mathcal{L}_{\vec{N}}\,{}^3\tilde{\Pi}^{rs} = \epsilon\Big[({}^3\tilde{\pi}^{rs}N^u)_{|u} - N^r{}_{|u}\,{}^3\tilde{\pi}^{us} - N^s{}_{|u}\,{}^3\tilde{\pi}^{ur}\Big],\\
&\Downarrow\\
\partial_\tau\,{}^3K_{rs}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big(N[{}^3R_{rs} + {}^3K\,{}^3K_{rs} - 2\,{}^3K_{ru}\,{}^3K^u{}_s] -\\
&- N_{|s|r} + N^u{}_{|s}\,{}^3K_{ur} + N^u{}_{|r}\,{}^3K_{us} + N^u\,{}^3K_{rs|u}\Big)(\tau,\vec{\sigma}),\\
\partial_\tau\,\gamma(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big(2\gamma[-N\,{}^3K + N^u{}_{|u}]\Big)(\tau,\vec{\sigma}),\\
\partial_\tau\,{}^3K(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big(N[{}^3g^{rs}\,{}^3R_{rs} + ({}^3K)^2] - N_{|u}{}^{|u} + N^u\,{}^3K_{|u}\Big)(\tau,\vec{\sigma}),
\end{aligned}
\tag{82}
$$

with

$\mathcal{L}_{\vec{N}}\,{}^3\tilde{\Pi}^{rs} = -\sqrt{\gamma}\,{}^3\nabla_u(\frac{N^u}{\sqrt{\gamma}}{}^3\tilde{\Pi}^{rs}) + {}^3\tilde{\Pi}^{ur}\,{}^3\nabla_u N^s + {}^3\tilde{\Pi}^{us}\,{}^3\nabla_u N^r.$

We have also used
$\delta(\sqrt{\gamma}\,{}^3R)(\tau,\vec{\sigma}) = \int d^3\sigma_1\{(\sqrt{\gamma}\,{}^3R)(\tau,\vec{\sigma}),{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}_1)\}\delta\,{}^3g_{rs}(\tau,\vec{\sigma}_1) = \int d^3\sigma_1\delta\,{}^3g_{rs}(\tau,\vec{\sigma}_1)\{[-\sqrt{\gamma}({}^3R^{rs} - \frac{1}{2}{}^3g^{rs}\,{}^3R)](\tau,\vec{\sigma})\delta^3(\vec{\sigma},\vec{\sigma}_1) + [\sqrt{\gamma}\,{}^3\Gamma^n_{lm}({}^3g^{rl}\,{}^3g^{sm} - {}^3g^{rs}\,{}^3g^{lm})](\tau,\vec{\sigma}_1)\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}_1)}{\partial\sigma^n} + [\sqrt{\gamma}({}^3g^{rl}\,{}^3g^{sm} - {}^3g^{rs}\,{}^3g^{lm})](\tau,\vec{\sigma}_1)\frac{\partial^2\delta^3(\vec{\sigma},\vec{\sigma}_1)}{\partial\sigma^l\partial\sigma^m}\}$.

The above equation for $\partial_\tau\,{}^3g_{rs}(\tau,\vec{\sigma})$ shows that the generator of space diffeomorphisms $\int d^3\sigma N_r(\tau,\vec{\sigma})\,{}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma})$ produces a variation, tangent to Σ_τ, $\delta_{tangent}{}^3g_{rs} = \mathcal{L}_{\vec{N}}\,{}^3g_{rs} = N_{r|s} + N_{s|r}$ in accord with the infinitesimal pseudodiffeomorphisms in $Diff\,\Sigma_\tau$. Instead, the superhamiltonian generator $\int d^3\sigma N(\tau,\vec{\sigma})\,\tilde{\mathcal{H}}(\tau,\vec{\sigma})$ does not reproduce the infinitesimal diffeomorphisms in $Diff\,M^4$ normal to Σ_τ (see also Ref. [48]). In Ref. [88] there is a study of the assumptions hidden in the ADM formulation (essentially the embedding of the model Σ hypersurface in M^4 is fixed and not variable), whose relaxation allows to turn an arbitrary normal deformation to Σ_τ (as an element of $Diff\,M^4$) into the deformation $-2N(\tau,\vec{\sigma})\,{}^3K_{rs}(\tau,\vec{\sigma})$ generated by the superhamiltonian constraint.

Let us remark that the canonical transformation [${}^4g_{AB}$ and ${}^4g^{AB}$ are given in Eqs.(6)] $\tilde{\Pi}^N\,dN + \tilde{\Pi}^r_{\vec{N}}\,dN_r + {}^3\tilde{\Pi}^{rs}\,d^3g_{rs} = {}^4\tilde{\Pi}^{AB}\,d^4g_{AB}$ defines the following momenta conjugated to ${}^4g_{AB}$

$$
\begin{aligned}
{}^4\tilde{\Pi}^{\tau\tau} &= \frac{\epsilon}{2N}\tilde{\Pi}^N,\\
{}^4\tilde{\Pi}^{\tau r} &= \frac{\epsilon}{2}(\frac{N^r}{N}\tilde{\Pi}^N - \tilde{\Pi}^r_{\vec{N}}),\\
{}^4\tilde{\Pi}^{rs} &= \epsilon(\frac{N^rN^s}{2N}\tilde{\Pi}^N - {}^3\tilde{\Pi}^{rs}),
\end{aligned}
$$

$$
\{{}^4g_{AB}(\tau,\vec{\sigma}),{}^4\tilde{\Pi}^{CD}(\tau,\vec{\sigma}^{'})\} = \frac{1}{2}(\delta^C_A\delta^D_B + \delta^D_A\delta^C_B)\delta^3(\vec{\sigma},\vec{\sigma}^{'}),
$$

$$
\begin{aligned}
\tilde{\Pi}^N &= \frac{2\epsilon}{\sqrt{\epsilon^4 g^{\tau\tau}}} {}^4\tilde{\Pi}^{\tau\tau},\\
\tilde{\Pi}^r_{\vec{N}} &= 2\epsilon \frac{{}^4g^{\tau r}}{{}^4g^{\tau\tau}} {}^4\tilde{\Pi}^{\tau\tau} - 2\epsilon {}^4\tilde{\Pi}^{\tau r},\\
{}^3\tilde{\Pi}^{rs} &= \epsilon \frac{{}^4g^{\tau r}{}^4g^{\tau S}}{({}^4g^{\tau\tau})^2}\, {}^4\tilde{\Pi}^{\tau\tau} - \epsilon {}^4\tilde{\Pi}^{rs},
\end{aligned}
\tag{83}
$$

which would emerge if the ADM action would be considered function of ${}^4g_{AB}$ instead of N, N_r and ${}^3g_{rs}$.

The standard ADM momenta ${}^3\tilde{\Pi}^{rs}$, defined in Eq. (72), may now be expressed in terms of the cotriads and their conjugate momenta of the canonical formulation of tetrad gravity given in Section IV:

$$
\begin{aligned}
{}^3\tilde{\Pi}^{rs} &= \epsilon k\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\,{}^3K) = \frac{1}{4}[{}^3e^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)} + {}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)}],\\
&\Rightarrow {}^3\tilde{\Pi} = {}^3\tilde{\Pi}^{rs}\,{}^3g_{rs} = -2\epsilon k\sqrt{\gamma}\,{}^3K = \frac{1}{2}{}^3e_{(a)r}\,{}^3\tilde{\pi}^r_{(a)},\\
&\{{}^3g_{rs}(\tau,\vec{\sigma}) = {}^3e_{(a)r}(\tau,\vec{\sigma})\,{}^3e_{(a)s}(\tau,\vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma}^{\,\prime})\} = \frac{1}{2}(\delta^u_r\delta^v_s + \delta^u_s\delta^v_r)\delta^3(\vec{\sigma},\vec{\sigma}^{\,\prime}),\\
&\{{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma}^{\,\prime})\} = \frac{1}{8}\delta^3(\vec{\sigma},\vec{\sigma}^{\,\prime}) \times\\
&[{}^3g^{ru}\,{}^3e^v_{(a)}\,{}^3e^s_{(b)} + {}^3g^{rv}\,{}^3e^u_{(a)}\,{}^3e^s_{(b)} + {}^3g^{su}\,{}^3e^v_{(a)}\,{}^3e^r_{(b)} + {}^3g^{sv}\,{}^3e^u_{(a)}\,{}^3e^r_{(b)}](\tau,\vec{\sigma}) \cdot\\
&\cdot {}^3\tilde{M}_{(a)(b)}(\tau,\vec{\sigma}) \approx 0.
\end{aligned}
\tag{84}
$$

The fact that in tetrad gravity the last Poisson brackets is only weakly zero has been noted in Ref. [67].

Let us now consider the expression of the ADM supermomentum constraints in tetrad gravity. Since ${}^3e_{(b)u}\,{}^3\tilde{\Pi}^{us} = \frac{1}{4}{}^3e_{(b)u}[{}^3e^u_{(a)}\,{}^3\tilde{\pi}^s_{(a)} + {}^3e^s_{(a)}\,{}^3\tilde{\pi}^u_{(a)}] = \frac{1}{4}[{}^3\tilde{\pi}^s_{(b)} + {}^3e^s_{(a)}\,{}^3e_{(b)u}\,{}^3\tilde{\pi}^u_{(a)}] = \frac{1}{4}[{}^3\tilde{\pi}^s_{(b)} + {}^3e^s_{(a)}({}^3e_{(a)u}\,{}^3\tilde{\pi}^u_{(b)} + {}^3\tilde{M}_{(b)(a)})] = \frac{1}{4}[2\,{}^3\tilde{\pi}^s_{(b)} - {}^3e^s_{(a)}\,{}^3\tilde{M}_{(a)(b)}]$, we have

$$
\begin{aligned}
&{}^3\tilde{\Pi}^{rs}{}_{|s} = \partial_s\,{}^3\tilde{\Pi}^{rs} + {}^3\Gamma^r_{su}\,{}^3\tilde{\Pi}^{us} =\\
&= \partial_s\,{}^3\tilde{\Pi}^{rs} + [\epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,{}^3\omega_{s(c)} - \partial_s\,{}^3e^r_{(b)}]{}^3e_{(b)u}\,{}^3\tilde{\Pi}^{us} =\\
&= \frac{1}{4}(\partial_s[{}^3e^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)} + {}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)}] -\\
&-[\epsilon_{(a)(c)(b)}\,{}^3e^r_{(a)}\,{}^3\omega_{s(c)} + \partial_s\,{}^3e^r_{(b)}] \cdot [2\,{}^3\tilde{\pi}^s_{(b)} - {}^3e^s_{(d)}\,{}^3\tilde{M}_{(d)(b)}]\,) =\\
&= \frac{1}{4}\{\,{}^3e^r_{(a)}[\partial_s\,{}^3\tilde{\pi}^s_{(a)} - 2\epsilon_{(a)(b)(c)}\,{}^3\omega_{s(b)}\,{}^3\tilde{\pi}^s_{(c)}] - {}^3\tilde{\pi}^s_{(a)}\partial_s\,{}^3e^r_{(a)} +\\
&+\partial_s({}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)}) + [\epsilon_{(a)(c)(b)}\,{}^3e^r_{(a)}\,{}^3\omega_{s(c)} + \partial_s\,{}^3e^r_{(b)}]{}^3e^s_{(d)}\,{}^3\tilde{M}_{(d)(b)}\,\} =\\
&= \frac{1}{4}\{2\,{}^3e^r_{(a)}\hat{\mathcal{H}}_{(a)} + \partial_s[{}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)} - {}^3e^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)}] -\\
&-[\epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,{}^3\omega_{s(b)} + \partial_s\,{}^3e^r_{(c)}]{}^3e^s_{(d)}\,{}^3\tilde{M}_{(c)(d)}\}.
\end{aligned}
\tag{85}
$$

Since ${}^3\tilde{\pi}^r_{(a)} = \frac{1}{2}{}^3e^r_{(b)}[{}^3e_{(b)u}\,{}^3\tilde{\pi}^u_{(a)} + {}^3e_{(a)u}\,{}^3\tilde{\pi}^u_{(b)}] - \frac{1}{2}{}^3\tilde{M}_{(a)(b)}\,{}^3e^r_{(b)}$, we get $\partial_s[{}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)} - {}^3e^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)}] = \partial_s[\frac{1}{2}({}^3e^s_{(a)}\,{}^3e^r_{(b)} - {}^3e^r_{(a)}\,{}^3e^s_{(b)})({}^3e_{(b)u}\,{}^3\tilde{\pi}^u_{(a)} + {}^3e_{(a)u}\,{}^3\tilde{\pi}^u_{(b)}) - {}^3e^s_{(a)}\,{}^3e^r_{(b)}\,{}^3\tilde{M}_{(a)(b)}] =$

$-\partial_s[{}^3e^s_{(a)}\,{}^3e^r_{(b)}\,{}^3\tilde{M}_{(a)(b)}]$, the ADM metric supermomentum constraints (79) are satisfied in the following form

$$
\begin{aligned}
{}^3\tilde{\mathcal{H}}^r &= -2\,{}^3\tilde{\Pi}^{rs}{}_{|s} = \frac{1}{2}\{\, -2\,{}^3e^r_{(a)}\hat{\mathcal{H}}_{(a)} + \partial_s[{}^3e^s_{(a)}\,{}^3e^r_{(b)}\,{}^3\tilde{M}_{(a)(b)}] + \\
&+ [\partial_s\,{}^3e^r_{(c)} - \epsilon_{(c)(b)(a)}\,{}^3\omega_{s(b)}\,{}^3e^r_{(a)}]\,{}^3e^s_{(d)}\,{}^3\tilde{M}_{(c)(d)}\,\} = \\
&= \frac{1}{2}\{\, 2\,{}^3e^r_{(a)}\,{}^3e^s_{(a)}\,{}^3\tilde{\Theta}_s + [{}^3e^r_{(a)}\,{}^3\omega_{s(b)} - {}^3e^r_{(b)}\,{}^3\omega_{s(a)}]\,{}^3e^s_{(a)}\,{}^3\tilde{M}_{(b)} + \\
&+ \epsilon_{(a)(b)(c)}\,{}^3e^r_{(b)}\partial_s[{}^3e^s_{(a)}\,{}^3\tilde{M}_{(c)}]\,\} \approx 0.
\end{aligned}
\tag{86}
$$

Let us add a comment on the structure of gauge-fixings for metric gravity; the same results hold for tetrad gravity. As said in Refs. [89,5], in a system with only primary and secondary first class constraints (like electromagnetism, Yang-Mills theory and both metric and tetrad gravity) the Dirac Hamiltonian H_D contains only the arbitrary Dirac multipliers associated with the primary first class constraints. The secondary first class constraints are already contained in the canonical Hamiltonian with well defined coefficients [the temporal components A_{ao} of the gauge potential in Yang-Mills theory; the lapse and shift functions in metric and tetrad gravity; in both cases, through the first half of the Hamilton equations, the Dirac multipliers turn out to be equal to the τ-derivatives of these quantities, which, therefore, inherit an induced arbitrariness]. See the second paper in Ref. [3] for a discussion of this point and for a refusal of Dirac's conjecture [1] according to which also the secondary first class constraints must have arbitrary Dirac multipliers (in such a case one does not recover the original Lagrangian by inverse Legendre transformation and one obtains a different "off-shell" theory). In these cases one must adopt the following gauge-fixing strategy: i) add gauge-fixing constraints $\chi_a \approx 0$ to the secondary constraints; ii) their time constancy, $\partial_\tau \chi_a \circeq \{\chi_a, H_D\} = g_a \approx 0$, implies the appearance of gauge-fixing constraints $g_a \approx 0$ for the primary constraints; iii) the time constancy of the constraints $g_a \approx 0$, $\partial_\tau g_a \circeq \{g_a, H_D\} \approx 0$, determines the Dirac multipliers in front of the primary constraints.

As shown in the second paper of Ref. [5] for the electromagnetic case, this method works also with covariant gauge-fixings: the electromagnetic Lorentz gauge $\partial^\mu A_\mu(x) \approx 0$ may be rewritten in phase space as a gauge-fixing constraint depending upon the Dirac multiplier; its time constancy gives a multiplier-dependent gauge-fixing for $A_o(x)$ and the time constancy of this new constraint gives the elliptic equation for the multiplier with the residual gauge freedom connected with the kernel of the elliptic operator.

In metric gravity, the covariant gauge-fixings analogous to the Lorentz gauge are those determining the harmonic coordinates (harmonic or DeDonder gauge): $\chi^B = \frac{1}{\sqrt{{}^4g}}\partial_A(\sqrt{{}^4g}\,{}^4g^{AB}) \approx 0$ in the Σ_τ-adapted holonomic coordinate basis. More explicitly, they are:

i) for $B=\tau$: $N\partial_\tau\gamma - \gamma\partial_\tau N - N^2\partial_r(\frac{\gamma N^r}{N}) \approx 0$;
ii) for $B=s$: $NN^s\partial_\tau\gamma + \gamma(N\partial_\tau N^s - N^s\partial_\tau N) + N^2\partial_r[N\gamma({}^3g^{rs} - \frac{N^rN^s}{N^2})] \approx 0$.

From Eqs.(82) we get $\partial_\tau N \circeq \lambda_N$, $\partial_\tau N_r \circeq \lambda^{\vec{N}}_r$ and $\partial_\tau\gamma = \frac{1}{2}\gamma\,{}^3g^{rs}\partial_\tau\,{}^3g_{rs} \circeq \frac{1}{2}\gamma[{}^3g^{rs}(N_{r|s}+N_{s|r}) - \frac{5\epsilon N}{k\sqrt{\gamma}}\,{}^3\tilde{\Pi}]$.

Therefore, in phase space the harmonic coordinate gauge-fixings take the form $\chi^B = \bar{\chi}^B(N, N_r, N_{r|s}, {}^3g_{rs}, {}^3\tilde{\Pi}^{rs}, \lambda_N, \lambda^{\vec{N}}_r) \approx 0$ and have to be associated with the secondary superhamiltonian and supermomentum constraints. The conditions $\partial_\tau \bar{\chi}^B \stackrel{\circ}{=} \{\bar{\chi}^B, H_D\} = g^B \approx 0$ give the gauge-fixings for the primary constraints $\tilde{\Pi}^N \approx 0$, $\tilde{\Pi}^r_{\vec{N}} \approx 0$. The conditions $\partial_\tau g^B \stackrel{\circ}{=} \{g^B, H_D\} \approx 0$ are partial differential equations for the Dirac multipliers λ_N, $\lambda^{\vec{N}}_r$, implying a residual gauge freedom like it happens for the electromagnetic Lorentz gauge.

VI. CONCLUSIONS.

Motivated by the attempt to get a unified description and a canonical reduction of the four interactions in the framework of Dirac-Bergmann theory of constraint (the presymplectic approach), we begin an investigation of general relativity along these lines. A complete analysis of this theory along these lines is still lacking, probably due to the fact that it does not respect the requirement of manifest general covariance. Instead, the presymplectic approach is the natural one to get an explicit control on the degrees of freedom of theories described by singular Lagrangians at the Hamiltonian level. After the completion of the canonical reduction along these lines, one will come back to the interpretational problems connected with general covariance, which are deeply different from those of ordinary gauge theories like Yang-Mills one.

In this first paper we have reviewed the kinematical framework for tetrad gravity (natural for the coupling to fermion fields) on globally hyperbolic, asymptotically flat at spatial infinity spacetimes whose 3+1 decomposition may be obtained with simultaneity spacelike hypersurfaces Σ_τ diffeomorphic to R^3 (they are the Cauchy surfaces).

Then, we have given a new parametrization of arbitrary cotetrads in terms of lapse and shift functions, of cotriads on Σ_τ and of three boost parameters. Such parametrized cotetrads are put in the ADM action for metric gravity to obtain the new Lagrangian for tetrad gravity. In the Hamiltonian formulation, we obtain 14 first class constraints, ten primary and four secondary ones, whose algebra is studied.

A comparison with other formulations of tetrad gravity and with the Hamiltonian ADM metric gravity has been done.

In the next paper [56], we shall study the Hamiltonian group of gauge transformations induced by the first class constraints. Then, the multitemporal equations associated with the constraints generating space rotations and space diffeomorphisms on the cotriads will be studied and solved. The Dirac observables with respect to thirteen of the fourteen constraints will be found in 3-orthogonal coordinates on Σ_τ and the associated Shanmugadhasan canonical transformation will be done. The only left constraint to be studied will be the superhamiltonian one. Some interpretational problems (Dirac observables versus general covariance) [90,26] will be faced.

APPENDIX A: 4-TENSORS IN THE Σ_τ-ADAPTED HOLONOMIC COORDINATES.

The connection coefficients ${}^4\Gamma^B_{AC} = \frac{1}{2}\,{}^4g^{BD}(\partial_A\,{}^4g_{CD} + \partial_C\,{}^4g_{AD} - \partial_D\,{}^4g_{AC}) = {}^4\Gamma^B_{CA}$ in the Σ_τ-adapted coordinate basis associated with ${}^4g_{AB}$ and ${}^4g^{AB}$ of Eqs.(46) and (47) are independent from the boost parameters $\varphi_{(a)}$ and have the following expression [use is made of $N_{v|r} = N_{(a)|r}\,{}^3e_{(a)v}$]

$$
\begin{aligned}
{}^4\Gamma^\tau_{\tau\tau} &= \frac{1}{N}\Big[\partial_\tau N + N^r\partial_r N - N^r N^s\,{}^3K_{rs}\Big] =\\
&= \frac{1}{N}\Big[\partial_\tau N + {}^3e^r_{(a)}N_{(a)}\partial_r N + \frac{N_{(a)}N_{(b)}}{N}\,{}^3e^r_{(a)}(\partial_\tau\,{}^3e_{(b)r} - N_{(b)|r})\Big],\\
{}^4\Gamma^\tau_{r\tau} &= {}^4\Gamma^\tau_{\tau r} = \frac{1}{N}\Big[\partial_r N - {}^3K_{rs}N^s\big] =\\
&= \frac{1}{N}[\partial_r N + \frac{N_{(a)}}{2N}(\delta^u_r\delta^v_s + \delta^u_s\delta^v_r)^3e^s_{(a)}\,{}^3e_{(b)u}(\partial_\tau\,{}^3e_{(b)v} - N_{(b)|v})\Big],\\
{}^4\Gamma^\tau_{rs} &= {}^4\Gamma^\tau_{sr} = -\frac{1}{N}{}^3K_{rs} =\\
&= \frac{1}{2N^2}(\delta^u_r\delta^v_s + \delta^u_s\delta^v_r)(\partial_\tau\,{}^3e_{(a)u} - N_{(a)|u})^3e_{(a)v},\\
{}^4\Gamma^u_{\tau\tau} &= \partial_\tau N^u - \frac{N^u}{N}\partial_\tau N + ({}^3g^{uv} - \frac{N^uN^v}{N^2})N\partial_v N + N^u{}_{|v}\,N^v -\\
&- 2N({}^3g^{uv} - \frac{N^uN^v}{2N^2})^3K_{vr}N^r =\\
&= {}^3e^u_{(a)}(\partial_\tau N_{(a)} - \frac{N_{(a)}}{N}\partial_\tau N) + N_{(a)}\partial_\tau\,{}^3e^u_{(a)} +\\
&+ N(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2})^3e^u_{(a)}\,{}^3e^v_{(b)}\partial_v N + {}^3e^v_{(b)}N_{(b)}({}^3e^u_{(a)}\,N_{(a)})||_{|v} -\\
&- (\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{2N^2})({}^3e^v_{(c)}\delta_{(b)(d)} + {}^3e^v_{(b)}\delta_{(c)(d)})\,{}^3e^u_{(a)}(N_{(d)|v} - \partial_\tau\,{}^3e_{(d)v})N_{(c)},\\
{}^4\Gamma^u_{r\tau} &= {}^4\Gamma^u_{\tau r} = N^u{}_{|r} - \frac{N^u}{N}\partial_r N - N({}^3g^{uv} - \frac{N^uN^v}{N^2})^3K_{vr} =\\
&= {}^3e^u_{(a)}(N_{(a)|r} - \frac{N_{(a)}}{N}\partial_r N) -\\
&- \frac{1}{2}\,{}^3e^u_{(a)}(\delta^u_r\delta_{(b)(c)} + {}^3e^s_{(b)}\,{}^3e_{(c)r})(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2})(N_{(c)|s} - \partial_\tau\,{}^3e_{(c)s}),\\
{}^4\Gamma^u_{rs} &= {}^4\Gamma^u_{sr} = {}^3\Gamma^u_{rs} - \frac{N^u}{N}{}^3K_{rs} =\\
&= {}^3\Gamma^u_{rs} + \frac{N_{(a)}}{2N^2}(\delta^m_r\delta^v_s + \delta^m_s\delta^v_r)^3e^u_{(a)}\,{}^3e_{(b)m}(\partial_\tau\,{}^3e_{(b)v} - N_{(b)|v}).
\end{aligned}
\tag{A1}
$$

In these equations we use the 3-dimensional Christoffel symbols ${}^3\Gamma^u_{rs}$, whose associated spin connection is ${}^3\omega_{r(a)(b)}$ of Eqs.(38).

The spacetime spin connection ${}^4\omega_{A(\alpha)(\beta)} = {}^4\eta_{(\alpha)(\gamma)}\,{}^4\omega_A{}^{(\gamma)}{}_{(\beta)}$

$$
\begin{aligned}
{}^4\omega_A{}^{(\alpha)}{}_{(\beta)} &= {}^4E^{(\alpha)}_B[\partial_A\,{}^4E^B_{(\beta)} + {}^4\Gamma^B_{AC}\,{}^4E^C_{(\beta)}] =\\
&= [\Lambda(\varphi_{(a)}(\sigma))\,{}^4\overset{\circ}{\omega}_A\,\Lambda^{-1}(\varphi_{(a)}(\sigma)) + \partial_A\Lambda(\varphi_{(a)}(\sigma))\,\Lambda^{-1}(\varphi_{(a)}(\sigma))]^{(\alpha)}{}_{(\beta)},
\end{aligned}
\tag{A2}
$$

is expressed in terms of the boost parameter independent spin connection [the Christoffel symbols are invariant under the local Lorentz rotation]

$$
{}^4\overset{\circ}{\omega}_A{}^{(\alpha)}{}_{(\beta)} = {}^4_{(\Sigma)}\check{\tilde{E}}^{(\alpha)}_B\, [\partial_A\, {}^4_{(\Sigma)}\check{\tilde{E}}^B_{(\beta)} + {}^4\Gamma^B_{AC}\, {}^4_{(\Sigma)}\check{\tilde{E}}^C_{(\beta)}]. \tag{A3}
$$

Analogously we have ${}^4\Omega_{AB}{}^{(\alpha)}{}_{(\beta)} = [\Lambda(\varphi_{(a)}(\sigma))\, {}^4\overset{\circ}{\Omega}_{AB}\, \Lambda^{-1}(\varphi_{(a)}(\sigma))]^{(\alpha)}{}_{(\beta)}$ for the associated field strengths.

For the spacetime spin connection ${}^4\overset{\circ}{\omega}_{A(\alpha)(\beta)} = {}^4\eta_{(\alpha)(\gamma)}\, {}^4\overset{\circ}{\omega}_A{}^{(\gamma)}{}_{(\beta)}$, also using Eqs.(50), (38) we have [see also Refs. [66]]

$$
\begin{aligned}
{}^4\overset{\circ}{\omega}_{\tau(o)(a)} &= -{}^4\overset{\circ}{\omega}_{\tau(a)(o)} = -\epsilon[\partial_r N + {}^3K_{rs}\, {}^3e^s_{(b)} N_{(b)}]^3 e^r_{(a)},\\
{}^4\overset{\circ}{\omega}_{\tau(a)(b)} &= -{}^4\overset{\circ}{\omega}_{\tau(b)(a)} =\\
&= -\epsilon\, {}^3e_{(a)r}\Big[\partial_\tau\, {}^3e^r_{(b)} - N_{(c)}\, {}^3e^s_{(c)}\partial_s\, {}^3e^r_{(b)} + {}^3e^s_{(b)}\partial_s(N_{(c)}\, {}^3e^r_{(c)})\Big] -\\
&- \epsilon[N\, {}^3e^r_{(a)}\, {}^3K_{rs}\, {}^3e^s_{(b)} + N_{(c)}\, {}^3e^r_{(c)}\, {}^3\omega_{r(a)(b)}] =\\
&= -\epsilon\, {}^3\omega_{r(a)(b)}\, {}^3e^r_{(c)} N_{(c)} - \frac{\epsilon}{2}\Big({}^3e^r_{(a)}\partial_\tau\, {}^3e_{(b)r} - {}^3e^r_{(b)}\partial_\tau\, {}^3e_{(a)r}\Big) +\\
&+ \frac{\epsilon}{2} N_{(c)}\, {}^3e^s_{(c)}\Big({}^3e_{(a)r}\partial_s\, {}^3e^r_{(b)} - {}^3e_{(b)r}\partial_s\, {}^3e^r_{(a)}\Big) -\\
&- \frac{\epsilon}{2}\Big({}^3e_{(a)s}\, {}^3e^r_{(b)} - {}^3e_{(b)s}\, {}^3e^r_{(a)}\Big)\partial_r(N_{(c)}\, {}^3e^s_{(c)}),\\
{}^4\overset{\circ}{\omega}_{r(o)(a)} &= -{}^4\overset{\circ}{\omega}_{r(a)(o)} = -\epsilon\, {}^3K_{rs}\, {}^3e^s_{(a)} =\\
&= -\frac{\epsilon}{2N}[\delta_{(a)(b)}\delta^w_r + {}^3e^w_{(a)}\, {}^3e_{(b)r}][N_{(b)|w} - \partial_\tau\, {}^3e_{(b)w}],\\
{}^4\overset{\circ}{\omega}_{r(a)(b)} &= -{}^4\overset{\circ}{\omega}_{r(b)(a)} = -\epsilon^3\omega_{r(a)(b)} = \frac{\epsilon}{2}\Big[{}^3e^s_{(a)}(\partial_r\, {}^3e_{(b)s} - \partial_s\, {}^3e_{(b)r}) +\\
&+ {}^3e^s_{(b)}(\partial_s\, {}^3e_{(a)r} - \partial_r\, {}^3e_{(a)s}) + {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^3e_{(c)r}(\partial_v\, {}^3e_{(c)u} - \partial_u\, {}^3e_{(c)v})\Big].
\end{aligned} \tag{A4}
$$

The field stregth ${}^4\overset{\circ}{\Omega}_{AB(\alpha)(\beta)} = {}^4_{(\Sigma)}\check{\tilde{E}}^C_{(\alpha)}\, {}^4_{(\Sigma)}\check{\tilde{E}}^D_{(\beta)}\, {}^4R_{CDAB} = \partial_A\, {}^4\overset{\circ}{\omega}_{B(\alpha)(\beta)} - \partial_B\, {}^4\overset{\circ}{\omega}_{A(\alpha)(\beta)} + {}^4\overset{\circ}{\omega}_{A(\alpha)(\gamma)}\, {}^4\overset{\circ}{\omega}^{(\gamma)}_{B(\beta)} - {}^4\overset{\circ}{\omega}_{B(\alpha)(\gamma)}\, {}^4\overset{\circ}{\omega}^{(\gamma)}_{A(\beta)}$ is obtained starting from the spin connection ${}^4\overset{\circ}{\omega}_{A(\alpha)(\beta)} = {}^4\eta_{(\alpha)(\gamma)}\, {}^4\overset{\circ}{\omega}^{(\gamma)}_{A(\beta)}$. We have [see Eqs.(38) for ${}^3\Omega_{rs(a)(b)}$]

$$
\begin{aligned}
{}^4\overset{\circ}{\Omega}_{rs(a)(b)} &= {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^4R_{uvrs} = -\epsilon\Big[{}^3\Omega_{rs(a)(b)} +\\
&+ ({}^3K_{ru}\, {}^3K_{sv} - {}^3K_{su}\, {}^3K_{rv})^3 e^u_{(a)}\, {}^3e^v_{(b)}\Big],\\
{}^4\overset{\circ}{\Omega}_{rs(o)(a)} &= \frac{1}{N}{}^3e^v_{(a)}({}^4R_{\tau vrs} - N^u\, {}^4R_{uvrs}) =\\
&= \epsilon({}^3K_{ru|s} - {}^3K_{su|r})\, {}^3e^u_{(a)},\\
{}^4\overset{\circ}{\Omega}_{\tau r(a)(b)} &= {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^4R_{uv\tau r} = -\epsilon\Big(\partial_\tau\, {}^3\omega_{r(a)(b)} +\\
&+ \frac{1}{2}(\epsilon_{(a)(b)(c)}\epsilon_{(d)(e)(f)} - \epsilon_{(a)(b)(d)}\epsilon_{(c)(e)(f)})\cdot\\
&\quad {}^3e^s_{(c)}\Big[\partial_\tau\, {}^3e_{(d)s} - \Big(N_{(g)}\, {}^3e^u_{(g)}\partial_u\, {}^3e_{(d)s} + {}^3e_{(d)u}\partial_s(N_{(g)}\, {}^3e^u_{(g)})\Big)\Big]\, {}^3\omega_{r(e)(f)} +\\
&+ N_{(c)}\, {}^3e^s_{(c)}[{}^3\omega_s, {}^3\omega_r]_{(a)(b)} +
\end{aligned}
$$

$$
\begin{aligned}
&+ {}^3K_{rs}\Big({}^3e^s_{(a)}\, {}^3e^u_{(b)} - {}^3e^u_{(a)}\, {}^3e^s_{(b)}\Big)\partial_u N + \\
&+ ({}^3K_{sv}\, {}^3K_{ru} - {}^3K_{uv}\, {}^3K_{rs}) {}^3e^u_{(a)}\, {}^3e^s_{(b)} N_{(c)}\, {}^3e^v_{(c)}\Big),\\
{}^4\overset{\circ}{\Omega}_{\tau r(o)(a)} &= \frac{1}{N}\, {}^3e^u_{(a)} ({}^4R_{\tau u\tau r} - N^s\, {}^4R_{su\tau r}) = -\epsilon\Big[\partial_\tau\, {}^3K_{rs} - \\
&- {}^3K_{ru}\, (N_{(b)}\, {}^3e^u_{(b)})_{|s} - {}^3K_{su}\, (N_{(b)}\, {}^3e^u_{(b)})_{|r} - N_{(b)}\, {}^3e^u_{(b)}\, {}^3K_{su|r} + N_{|s|r}\Big] {}^3e^s_{(a)}. \qquad \text{(A5)}
\end{aligned}
$$

The Riemann tensor ${}^4R_{ABCD} = {}^4_{(\Sigma)}\check{\tilde{E}}^{(\alpha)}_C\, {}^4_{(\Sigma)}\check{\tilde{E}}^{(\beta)}_D\, {}^4\overset{\circ}{\Omega}_{AB(\alpha)(\beta)} = {}^4g_{AE}\, {}^4R^E{}_{BCD} = -{}^4R_{ABDC} = -{}^4R_{BACD} = {}^4R_{CDAB} = \frac{1}{2}(\partial_B\partial_D\, {}^4g_{AC} + \partial_A\partial_C\, {}^4g_{BD} - \partial_A\partial_D\, {}^4g_{BC} - \partial_B\partial_C\, {}^4g_{AD}) + {}^4g_{EF}({}^4\Gamma^E_{AC}\, {}^4\Gamma^F_{BD} - {}^4\Gamma^E_{AD}\, {}^4\Gamma^F_{BC})$ has the following expression in the new basis

$$
\begin{aligned}
{}^4R_{rsuv} &= -\epsilon[{}^3R_{rsuv} + {}^3K_{ru}\, {}^3K_{sv} - {}^3K_{rv}\, {}^3K_{su}] = \\
&= {}^3e_{(a)r}\, {}^3e_{(b)s}\, {}^4\overset{\circ}{\Omega}_{uv(a)(b)} = \\
&= -\epsilon\Big[{}^3e_{(a)r}\, {}^3e_{(b)s}\, {}^3\Omega_{uv(a)(b)} + {}^3K_{ru}\, {}^3K_{sv} - {}^3K_{rv}\, {}^3K_{su}\Big],\\
{}^4R_{\tau ruv} &= {}^3e_{(a)u}\, {}^3e_{(b)v}\, {}^4\overset{\circ}{\Omega}_{\tau r(a)(b)} = N\, {}^3e_{(a)r}\, {}^4\overset{\circ}{\Omega}_{uv(o)(a)} + N_{(a)}\, {}^3e_{(b)r}\, {}^4\overset{\circ}{\Omega}_{uv(a)(b)} = \\
&= \epsilon\Big[N({}^3K_{ur|v} - {}^3K_{vr|u}) - N_{(a)}({}^3e_{(b)r}\, {}^3\Omega_{uv(a)(b)} + ({}^3K_{rv}\, {}^3K_{us} - {}^3K_{ru}\, {}^3K_{sv}) {}^3e^s_{(a)})\Big] = \\
&= -\frac{\epsilon}{2}\Big[\partial_\tau(\partial_v\, {}^3g_{ru} - \partial_u\, {}^3g_{rv}) - \partial_r\Big(\partial_u({}^3e_{(a)v}N_{(a)}) - \partial_v({}^3e_{(a)u}N_{(a)})\Big)\Big] + \\
&+ \epsilon\Big[(N^2 - N_{(a)}N_{(a)})({}^4\Gamma^\tau_{\tau v}\, {}^4\Gamma^\tau_{ru} - {}^4\Gamma^\tau_{\tau u}\, {}^4\Gamma^\tau_{rv}) - {}^3g_{mn}({}^4\Gamma^m_{\tau v}\, {}^4\Gamma^n_{ru} - {}^4\Gamma^m_{\tau u}\, {}^4\Gamma^n_{rv}) - \\
&- {}^3e_{(a)m}N_{(a)}({}^4\Gamma^\tau_{\tau v}\, {}^4\Gamma^m_{ru} + {}^4\Gamma^m_{\tau v}\, {}^4\Gamma^\tau_{ru} - {}^4\Gamma^\tau_{\tau u}\, {}^4\Gamma^m_{rv} - {}^4\Gamma^m_{\tau u}\, {}^4\Gamma^\tau_{rv})\Big],\\
{}^4R_{\tau r\tau s} &= N\, {}^3e_{(a)r}\, {}^4\overset{\circ}{\Omega}_{\tau s(o)(a)} + N_{(a)}\, {}^3e_{(b)r}\, {}^4\overset{\circ}{\Omega}_{\tau s(a)(b)} = -\epsilon\Big(N\, \Big[\partial_\tau\, {}^3K_{rs} - \\
&- {}^3K_{su}(N_{(a)}\, {}^3e^u_{(a)})_{|r} - {}^3K_{ru}(N_{(a)}\, {}^3e^u_{(a)})_{|s} - N_{(a)}\, {}^3e^u_{(a)}\, {}^3K_{ru|s} + N_{|r|s}\Big] + \\
&+ N_{(a)}\, \Big[{}^3e_{(b)r}\, \partial_\tau\, {}^3\omega_{s(a)(b)} + \frac{1}{2}(\epsilon_{(a)(b)(c)}\epsilon_{(d)(e)(f)} - \epsilon_{(a)(b)(d)}\epsilon_{(c)(e)(f)}) {}^3e_{(b)r}\, {}^3e^w_{(e)}\, \cdot \\
&\cdot\ \Big[\partial_\tau\, {}^3e_{(d)w} - (N_{(g)}\, {}^3e^u_{(g)}\partial_u\, {}^3e_{(d)w} + {}^3e_{(d)u}\, \partial_w(N_{(g)}\, {}^3e^u_{(g)}))\Big] {}^3\omega_{s(e)(f)} + \\
&+ N_{(c)}\, {}^3e^w_{(c)}\, {}^3e_{(b)r}[{}^3\omega_w, {}^3\omega_s]_{(a)(b)} + {}^3K_{sw}(\delta^u_r\, {}^3e^w_{(a)} - \delta^w_r\, {}^3e^u_{(a)})\partial_u N + \\
&+ ({}^3K_{rv}\, {}^3K_{su} - {}^3K_{uv}\, {}^3K_{rs}) {}^3e^u_{(a)} N_{(c)}\, {}^3e^v_{(c)}\Big]\Big) = \\
&= -\frac{\epsilon}{2}\Big[-\partial^2_\tau\, {}^3g_{rs} + \partial_\tau\Big(\partial_s({}^3e_{(a)r}N_{(a)}) + \partial_r({}^3e_{(a)s}N_{(a)})\Big) - \partial_r\partial_s(N^2 - N_{(a)}N_{(a)})\Big] + \\
&+ \epsilon\Big[(N^2 - N_{(a)}N_{(a)})({}^4\Gamma^\tau_{\tau s}\, {}^4\Gamma^\tau_{\tau r} - {}^4\Gamma^\tau_{\tau\tau}\, {}^4\Gamma^\tau_{rs}) - {}^3g_{mn}({}^4\Gamma^m_{\tau s}\, {}^4\Gamma^n_{\tau r} - {}^4\Gamma^m_{\tau\tau}\, {}^4\Gamma^n_{rs}) - \\
&- {}^3e_{(a)m}N_{(a)}({}^4\Gamma^\tau_{\tau s}\, {}^4\Gamma^m_{\tau r} + {}^4\Gamma^m_{\tau s}\, {}^4\Gamma^\tau_{\tau r} - {}^4\Gamma^\tau_{\tau\tau}\, {}^4\Gamma^m_{rs} - {}^4\Gamma^m_{\tau\tau}\, {}^4\Gamma^\tau_{rs})\Big]. \qquad \text{(A6)}
\end{aligned}
$$

While the expression of ${}^4R_{rsuv}$ in the holonomic basis coincides with the Gauss equation (10) in the nonholonomic basis, the expressions of ${}^4R_{\tau ruv}$ and ${}^4R_{\tau r\tau s}$ are the analogue in the holonomic basis of the Codazzi-Mainardi and Ricci equations respectively for ${}^4\bar{R}_{lruv}$ and ${}^4\bar{R}_{lrls}$ in the nonholonomic basis. Moreover, we have [$\overset{\circ}{=}$ refers to the use of vacuum Einstein equations]

$$
{}^4R_{AB} = {}^4R_{BA} = {}^4g^{CD}\, {}^4R_{CADB} =
$$

$$= \frac{\epsilon}{N^2}\,{}^4R_{\tau A\tau B} - \frac{\epsilon N^r}{N^2}({}^4R_{\tau ArB} + {}^4R_{rA\tau B}) - \epsilon({}^3g^{rs} - \frac{N^rN^s}{N^2})\,{}^4R_{rAsB} \stackrel{\circ}{=} 0,$$

$${}^4R_{\tau\tau} = -\epsilon{}^3e^r_{(a)}\,{}^3e^s_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2})\,{}^4R_{r\tau s\tau} \stackrel{\circ}{=} 0,$$

$${}^4R_{\tau u} = {}^4R_{u\tau} = \frac{\epsilon{}^3e^v_{(a)}N_{(a)}}{N^2}\,{}^4R_{\tau u\tau v} - \epsilon{}^3e^r_{(a)}\,{}^3e^v_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2})\,{}^4R_{\tau ruv} \stackrel{\circ}{=} 0,$$

$${}^4R_{rs} = {}^4R_{sr} = \frac{\epsilon}{N^2}\,{}^4R_{\tau r\tau s} - \frac{\epsilon{}^3e^u_{(a)}N_{(a)}}{N^2}({}^4R_{\tau rus} + {}^4R_{\tau sur}) -$$
$$- \epsilon{}^3e^u_{(a)}\,{}^3e^v_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2})\,{}^4R_{urvs} \stackrel{\circ}{=} 0,$$

$${}^4R = {}^4g^{AB}\,{}^4R_{AB} = \frac{\epsilon}{N^2}\,{}^4R_{\tau\tau} - 2\frac{\epsilon{}^3e^u_{(a)}N_{(a)}}{N^2}\,{}^4R_{\tau u} - \epsilon{}^3e^r_{(a)}\,{}^3e^s_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2})\,{}^4R_{rs} =$$
$$= -\frac{2}{N^2}{}^3e^r_{(a)}\,{}^3e^s_{(a)}\,{}^4R_{\tau r\tau s} + 4\frac{{}^3e^u_{(c)}N_{(c)}}{N^2}{}^3e^r_{(a)}\,{}^3e^v_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2}){}^4R_{\tau ruv} +$$
$$+ {}^3e^r_{(a)}\,{}^3e_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2}){}^3e^u_{(c)}\,{}^3e^v_{(d)}(\delta_{(c)(d)} - \frac{N_{(c)}N_{(d)}}{N^2}){}^4R_{rusv} \stackrel{\circ}{=} 0, \qquad \text{(A7)}$$

$${}^4C_{ABCD} = {}^4R_{ABCD} + \frac{1}{2}({}^4R_{AC}\,{}^4g_{BD} - {}^4R_{BC}\,{}^4g_{AD} + {}^4R_{BD}\,{}^4g_{AC} - {}^4R_{AD}\,{}^4g_{BC}) +$$
$$+ \frac{1}{6}({}^4g_{AC}\,{}^4g_{BD} - {}^4g_{AD}\,{}^4g_{BC})\,{}^4R \stackrel{\circ}{=} {}^4R_{ABCD},$$

$${}^4C_{rsuv} = {}^4R_{rsuv} + \frac{\epsilon}{2}({}^3g_{rv}\,{}^4R_{su} + {}^3g_{su}\,{}^4R_{rv} - {}^3g_{ru}\,{}^4R_{sv} - {}^3g_{sv}\,{}^4R_{ru}) +$$
$$+ \frac{1}{6}({}^3g_{ru}\,{}^3g_{sv} - {}^3g_{rv}\,{}^3g_{su})\,{}^4R =$$
$$= {}^4R_{rsuv} - \frac{1}{2}{}^3e^m_{(a)}\,{}^3e^n_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2}) \times$$
$$\Big[{}^3g_{rv}\,{}^4R_{msnu} + {}^3g_{su}\,{}^4R_{mrnv} - {}^3g_{ru}\,{}^4R_{msnv} - {}^3g_{sv}\,{}^4R_{mrnu}\Big] +$$
$$+ \frac{1}{6}({}^3g_{ru}\,{}^3g_{sv} - {}^3g_{rv}\,{}^3g_{su}){}^3e^m_{(a)}\,{}^3e^n_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2}) \times$$
$${}^3e^w_{(c)}\,{}^3e^t_{(d)}(\delta_{(c)(d)} - \frac{N_{(c)}N_{(d)}}{N^2}){}^4R_{mwnt} +$$
$$+ \frac{1}{2N^2}\Big({}^3g_{rv}\,{}^4R_{\tau s\tau u} + {}^3g_{su}\,{}^4R_{\tau r\tau u} - {}^3g_{ru}\,{}^4R_{\tau s\tau v} - {}^3g_{sv}\,{}^4R_{\tau r\tau u}\Big) -$$
$$- \frac{1}{3N^2}({}^3g_{ru}\,{}^3g_{sv} - {}^3g_{rv}\,{}^3g_{su}){}^3e^m_{(a)}\,{}^3e^n_{(a)}\,{}^4R_{\tau m\tau n} -$$
$$- \frac{1}{2N^2}{}^3e^m_{(a)}N_{(a)}\Big[{}^3g_{rv}({}^4R_{\tau smu} + {}^4R_{\tau ums}) + {}^3g_{su}({}^4R_{\tau rmv} + {}^4R_{\tau vmr}) -$$
$$- {}^3g_{ru}({}^4R_{\tau smv} + {}^4R_{\tau vms}) - {}^3g_{sv}({}^4R_{\tau rmu} + {}^4R_{\tau umr})\Big] +$$
$$+ \frac{2}{3N^2}{}^3e^m_{(c)}N_{(c)}({}^3g_{ru}\,{}^3g_{sv} - {}^3g_{rv}\,{}^3g_{su}){}^3e^w_{(a)}\,{}^3e^n_{(b)}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2}){}^4R_{\tau wmn} \stackrel{\circ}{=}$$
$$\stackrel{\circ}{=} {}^4R_{rsuv},$$

$${}^4C_{\tau ruv} = {}^4R_{\tau ruv} + \frac{\epsilon}{2}[{}^3g_{ru}\,{}^4R_{\tau v} - {}^3g_{rv}\,{}^4R_{\tau u} + N_{(a)}({}^3e_{(a)v}\,{}^4R_{ru} - {}^3e_{(a)u}\,{}^4R_{rv})] +$$

$$
\begin{aligned}
&+ \frac{1}{6} N_{(a)} ({}^3e_{(a)u}\, {}^3g_{rv} - {}^3e_{(a)v}\, {}^3g_{ru})^4R = \\
&= {}^4R_{\tau ruv} - \frac{1}{2}\Big({}^3e^m_{(a)}\, {}^3e^n_{(b)} (\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2})({}^3g_{ru}\, {}^4R_{\tau mvn} - {}^3g_{rv}\, {}^4R_{\tau mun}) - \\
&- \frac{N_{(a)}\, {}^3e^m_{(b)} N_{(b)}}{N^2}\Big[{}^3e_{(a)v} ({}^4R_{\tau rmu} + {}^4R_{\tau umr}) - {}^3e_{(a)u} ({}^4R_{\tau rmv} + {}^4R_{\tau vmr})\Big]\Big) + \\
&+ \frac{2}{3N^2} N_{(a)} ({}^3e_{(a)u}\, {}^3g_{rv} - {}^3e_{(a)v}\, {}^3g_{ru})^3e^m_{(b)} N_{(b)}{}^3e^w_{(c)}\, {}^3e^n_{(d)} (\delta_{(c)(d)} - \frac{N_{(c)}N_{(d)}}{N^2})^4r_{\tau wmn} + \\
&+ \frac{N_{(a)}}{2N^2}\Big[{}^3e^m_{(a)} ({}^3g_{ru}\, {}^4R_{\tau v\tau m} - {}^3g_{rv}\, {}^4R_{\tau u\tau m}) + {}^3e_{(a)v}\, {}^4R_{\tau r\tau u} - {}^3e_{(a)u}\, {}^4R_{\tau r\tau v}\Big] - \\
&- \frac{N_{(a)}}{3N^2} ({}^3e_{(a)u}\, {}^3g_{rv} - {}^3e_{(a)v}\, {}^3g_{ru})^3e^m_{(b)}\, {}^3e^n_{(b)}\, {}^4R_{\tau m\tau n} - \\
&- \frac{1}{2} N_{(a)}\, {}^3e^m_{(b)}\, {}^3e^n_{(c)} (\delta_{(b)(c)} - \frac{N_{(b)}N_{(c)}}{N^2})({}^3e_{(a)v}\, {}^4R_{mrnu} - {}^3e_{(a)u}\, {}^4R_{mrnv}) + \\
&+ \frac{1}{6} N_{(a)} ({}^3e_{(a)u}\, {}^3g_{rv} - {}^3e_{(a)v}\, {}^3g_{ru})^3e^m_{(b)}\, {}^3e^n_{(c)} (\delta_{(b)(c)} - \frac{N_{(b)}N_{(c)}}{N^2}) \times \\
&\quad {}^3e^w_{(d)}\, {}^3e^t_{(e)} (\delta_{(d)(e)} - \frac{N_{(d)}N_{(e)}}{N^2})^4r_{mnwt} \,\hat{=}\, {}^4R_{\tau ruv}, \\
{}^4C_{\tau r\tau s} &= {}^4R_{\tau r\tau s} + \frac{\epsilon}{2}[N_{(a)}({}^3e_{(a)s}\, {}^4R_{\tau r} + {}^3e_{(a)r}\, {}^4R_{\tau s}) - {}^3g_{rs}\, {}^4R_{\tau\tau} + \\
&+ (N^2 - N_{(a)}N_{(a)})\, {}^4R_{rs}] - \frac{1}{6}[{}^3g_{rs}\, (N^2 - N_{(a)}N_{(a)}) + N_{(a)}N_{(b)}{}^3e_{(a)r}\, {}^3e_{(b)s}\,]\, {}^4R = \\
&= (2 - \frac{N_{(a)}N_{(a)}}{N^2})^4R_{\tau r\tau s} + \frac{1}{2}\Big(\frac{N_{(a)}\, {}^3e^m_{(b)} N_{(b)}}{N^2} ({}^3e_{(a)s}\, {}^4R_{\tau r\tau m} + {}^3e_{(a)r}\, {}^4R_{\tau s\tau m}) + \\
&+ \frac{1}{3}\Big[{}^3g_{rs} (1 - \frac{N_{(a)}N_{(a)}}{N^2}) + \frac{N_{(a)}N_{(b)}}{N^2} {}^3e_{(a)r}\, {}^3e_{(b)s}\Big]^3e^m_{(c)}\, {}^3e^n_{(c)}\, {}^3R_{\tau m\tau n}\Big) - \\
&- \frac{1}{2}\Big[N_{(a)}\, {}^3e^m_{(b)}\, {}^3e^n_{(c)} (\delta_{(b)(c)} - \frac{N_{(b)}N_{(c)}}{N^2})({}^3e_{(a)s}\, {}^4R_{\tau mrn} + {}^3e_{(a)r}\, {}^4R_{\tau msn}) + \\
&+ (1 - \frac{N_{(a)}N_{(a)}}{N^2})^3e^u_{(b)} N_{(b)} ({}^4R_{\tau rus} + {}^4R_{\tau sur})\Big] - \\
&- \frac{2}{3}[{}^3g_{rs} (1 - \frac{N_{(a)}N_{(a)}}{N^2}) + \frac{N_{(a)}N_{(b)}}{N^2} {}^3e_{(a)r}\, {}^3e_{(b)s}]^3e^u_{(c)} N_{(c)} \\
&\quad {}^3e^m_{(d)}\, {}^3e^n_{(e)} (\delta_{(d)(e)} - \frac{N_{(d)}N_{(e)}}{N^2})^4R_{\tau mun} + \\
&+ (N^2 - N_{(a)}N_{(a)})^3e^m_{(b)}\, {}^3e^n_{(c)} (\delta_{(b)(c)} - \frac{N_{(b)}N_{(c)}}{N^2})^4R_{mrns} - \\
&- \frac{1}{6}[{}^3g_{rs} (N^2 - N_{(a)}N_{(a)}) + N_{(a)}N_{(b)}\, {}^3e_{(a)r}\, {}^3e_{(b)s}]^3e^m_{(c)}\, {}^3e^n_{(d)} (\delta_{(c)(d)} - \frac{N_{(c)}N_{(d)}}{N^2}) \\
&\quad {}^3e^w_{(e)}\, {}^3e^t_{(f)} (\delta_{(e)(f)} - \frac{N_{(e)}N_{(f)}}{N^2})^4R_{mwnt} \,\hat{=}\, {}^4R_{\tau r\tau s}. \qquad \text{(A8)}
\end{aligned}
$$

so that every quantity can be expressed in terms of ${}^4R_{rsuv}$, ${}^4R_{\tau ruv}$, ${}^4R_{\tau r\tau s}$.

For the electric ${}^4E_{AB} = {}^4C_{A\tau B\tau}$ and magnetic ${}^4H_{AB} = \frac{1}{2}\epsilon_{B\tau EF}\, {}^4C_{A\tau}{}^{EF}$ components of the Weyl tensor (by assuming that the normals to Σ_τ are the privileged timelike 4-vectors) we have ${}^4E_{\tau\tau} = {}^4E_{\tau r} = {}^4H_{\tau\tau} = {}^4H_{\tau r} = 0$, ${}^4E_{rs} = {}^4C_{r\tau s\tau} \,\hat{=}\, {}^4R_{r\tau s\tau}$ and ${}^4H_{rs} = \frac{1}{2}\epsilon_{s\tau}{}^{uv}\, {}^4C_{r\tau uv} \,\hat{=}\, \frac{1}{2}\epsilon_{s\tau}{}^{uv}\, {}^4R_{r\tau uv}$.

In the coordinates $\sigma^A = \{\tau, \vec{\sigma}\}$ the "geodesic equation" is $\frac{d^2\sigma^A(s)}{ds^2} + {}^4\Gamma^A_{BC} \frac{d\sigma^B(s)}{ds} \frac{d\sigma^C(s)}{ds} = 0$, while the "geodesic deviation equation" is $a^A = \frac{d\sigma^B}{ds}\, {}^4\nabla_B (\frac{d\sigma^C}{ds}\, {}^4\nabla_C \triangle x^A) = -{}^4R^A{}_{BCD} \triangle x^C \frac{d\sigma^B}{ds} \frac{d\sigma^D}{ds}$. The results of this Appendix allow the identification of the dependence of 4-geodesics and of 4-geodesic deviations on the gauge parameters of the theory $[\sigma^A(s) = (\tau(s); \vec{\sigma}(s))]$:

$$\begin{aligned}
&\frac{d^2\tau(s)}{ds^2} + {}^4\Gamma^\tau_{\tau\tau} (\frac{d\tau(s)}{ds})^2 + 2\, {}^4\Gamma^\tau_{\tau r} \frac{d\tau(s)}{ds} \frac{d\sigma^r(s)}{ds} + {}^4\Gamma^\tau_{rs} \frac{d\sigma^r(s)}{ds} \frac{d\sigma^s(s)}{ds} = 0,\\
&\frac{d^2\sigma^u(s)}{ds^2} + {}^4\Gamma^u_{\tau\tau} (\frac{d\tau(s)}{ds})^2 + 2\, {}^4\Gamma^u_{\tau r} \frac{d\tau(s)}{ds} \frac{d\sigma^r(s)}{ds} + {}^4\Gamma^u_{rs} \frac{d\sigma^r(s)}{ds} \frac{d\sigma^s(s)}{ds} = 0,
\end{aligned}$$

$$\begin{aligned}
a^\tau &= -\frac{\epsilon}{N^2} \Big(\Big[{}^4R_{\tau m\tau n} \frac{d\sigma^m}{ds} \frac{d\sigma^n}{ds} - {}^3e^r_{(a)} N_{(a)} (-{}^4R_{\tau r\tau n} \frac{d\tau}{ds} + {}^4R_{rm\tau n} \frac{d\sigma^m}{ds}) \frac{d\sigma^n}{ds} \Big] \triangle x^\tau +\\
&+ \Big[(-{}^4R_{\tau m\tau s} \frac{d\tau}{ds} + {}^4R_{\tau msn} \frac{d\sigma^n}{ds}) \frac{d\sigma^m}{ds} - {}^3e^r_{(a)} N_{(a)} \Big({}^4R_{\tau r\tau s} (\frac{d\tau}{ds})^2 -\\
&- ({}^4R_{\tau rsm} + {}^4R_{rm\tau s}) \frac{d\tau}{ds} \frac{d\sigma^m}{ds} + {}^4R_{rmsn} \frac{d\sigma^m}{ds} \frac{d\sigma^s}{ds} \Big) \Big] \triangle x^s \Big),\\
a^u &= \epsilon \Big(\Big[\frac{{}^3e^u_{(a)} N_{(a)}}{N^2} {}^4R_{\tau m\tau n} \frac{d\sigma^m}{ds} \frac{d\sigma^n}{ds} + {}^3e^u_{(a)}\, {}^3e^r_{(b)} (\delta_{(a)(b)} - \frac{N_{(a)} N_{(b)}}{N^2}) \times\\
&(-{}^4R_{\tau r\tau n} \frac{d\tau}{ds} + {}^4R_{rm\tau n} \frac{d\sigma^m}{ds}) \frac{d\sigma^n}{ds} \Big] \triangle x^\tau +\\
&+ \Big[\frac{{}^3e^u_{(a)} N_{(a)}}{N^2} (-{}^4R_{\tau m\tau s} \frac{d\tau}{ds} + {}^4R_{\tau msn} \frac{d\sigma^n}{ds}) \frac{d\sigma^m}{ds} +\\
&+ {}^3e^u_{(a)}\, {}^3e^r_{(b)} (\delta_{(a)(b)} - \frac{N_{(a)} N_{(b)}}{N^2}) \Big({}^4R_{\tau r\tau s} (\frac{d\tau}{ds})^2 -\\
&- ({}^4R_{\tau rsm} + {}^4R_{rm\tau s}) \frac{d\tau}{ds} \frac{d\sigma^m}{ds} + {}^4R_{rmsn} \frac{d\sigma^m}{ds} \frac{d\sigma^n}{ds} \Big) \Big] \triangle x^s \Big). \qquad \text{(A9)}
\end{aligned}$$

More in general, to describe an arbitrary (not necessarily geodetic) congruence of timelike curves (congruence of observers; it is surface forming in absence of vorticity) with tangent field $u^A = {}^4E^A_{(o)}[\varphi_{(a)}, N, N_{(a)}, {}^3e^r_{(a)}]$ (see Eq.(47); by varying the 3 functions $\varphi_{(a)}(\tau, \vec{\sigma})$ we can describe any congruence) one uses [see for instance Ref. [91], where there is a reformulation of Newman-Penrose formalism replacing the congruence of lightlike curves with one of timelike ones; for the "threading" viewpoint (3+1 decomposition with respect to an arbitrary timelike congruence) see also Refs. [92,93]]

$$\begin{aligned}
{}^4\nabla_A u_B &= \epsilon u_A \dot{u}_B + \sigma_{AB} + \frac{1}{3}\Theta ({}^4g_{AB} - \epsilon u_A u_B) - \omega_{AB},\\
&\dot{u}^A = u^B\, {}^4\nabla_B u^A, \quad acceleration,\\
&\Theta = {}^4\nabla_A u^A, \quad (volume)\, rate\, of\, expansion\, scalar,\\
&\sigma_{AB} = \sigma_{BA} = -\epsilon \dot{u}_{(A} u_{B)} + {}^4\nabla_{(A} u_{B)} - \frac{1}{3}\Theta ({}^4g_{AB} - \epsilon u_A u_B),\\
&rate\, of\, shear\, tensor\, (with\, magnitude\, \sigma^2 = \frac{1}{2}\sigma_{AB}\sigma^{AB}),\\
&\omega_{AB} = -\omega_{BA} = \epsilon_{ABCD}\omega^C u^D = -u_{[A}\dot{u}_{B]} - {}^4\nabla_{[A} u_{B]}, \quad twist\, or\, vorticity\, tensor,\\
&\omega^A = \frac{1}{2}\epsilon^{ABCD}\omega_{BC} u_D, \quad vorticity\, vector. \qquad \text{(A10)}
\end{aligned}$$

Associated quantities are: i) the representative length l along the worldlines of u^A, describing the volume expansion (contraction) behaviour of the congruence completely, by the equation $\frac{1}{l}\, u^A\, {}^4\nabla_A l = \frac{1}{3}\Theta$; ii) the Hubble parameter H: $H = \frac{1}{l}\, u^A\, {}^4\nabla_A l = \frac{1}{3}\Theta$; iii) the dimensionless (cosmological) decelaration parameter: $q = -\frac{l\, u^A\, {}^4\nabla_A (u^B\, {}^4\nabla_B l)}{(u^D\, {}^4\nabla_D l)^2} = 3 u^A\, {}^4\nabla_A \frac{1}{\Theta} - 1$.

If the congruence is geodesic, the geodesic deviation equation yields equations for the rate of change of Θ, σ_{AB} and ω_{AB} along each geodesic in the congruence [see Ref. [38] for both timelike and null congruences; the equation for Θ is the Raychauduri equation].

Let $\sigma^A(s) = \{\tau(s); \vec{\sigma}(s)\}$ be a timelike geodesic Γ with timelike tangent vector $u^A(s) = \frac{d\sigma^A(s)}{ds}$, $u^A(s) u_A(s) = \epsilon$, $u^B(s)\, {}^4\nabla_B u^A(s) = 0$ [the affine parameter s is the proper time]. Let us consider a tetrad field ${}^4E^A_{(\alpha)}(\tau, \vec{\sigma})[\varphi_{(a)}, N, N_{(a)}, {}^3e^r_{(a)}]$, whose restriction to the geodesic Γ has ${}^4E^A_{(o)}(\sigma(s)) = u^A(s)$ [many tetrad fields satisfy this requirement: they differ in the space axes ${}^4E^A_{(a)}(\sigma(s))$]: the tetrad ${}^4E^A_{(\alpha)}(\sigma(s))$ describes an accelerated observer with worldline Γ. By going to Riemann normal coordinates for M^4 [they are not uniquely determined: see Appendix A of Ref. [56] for a review; in them we have at the point $\sigma^A(s)$ [71]:
${}^4g_{AB} = {}^3\eta_{AB}$, $\partial_C\, {}^4g_{AB} = 0$, ${}^4\Gamma^A_{BC} = 0$, $\partial_C \partial_D\, {}^4g_{AB} = -\frac{1}{3}({}^4R_{ACBD} + {}^4R_{ADBC}) = -\frac{2}{3}\, {}^4J_{ABCD}$ (${}^4J$ is the Jacobi curvature tensor,carrying the same information of the Riemann tensor), $\partial_D\, {}^4\Gamma^A_{BC} = -\frac{1}{3}({}^4R^A{}_{BCD} + {}^4R^A{}_{CBD})$, ${}^4R_{ABCD} = \partial_B \partial_C\, {}^4g_{AD} - \partial_B \partial_D\, {}^4g_{AC}$]
such that the timelike geodesic Γ becomes a timelike straightline, we get the description of a "comoving inertial frame" for an observer in free fall at rest: by a suitable choice of the gauge parameters $\varphi_{(a)}$, N, $N_{(a)}$, ..., along Γ we can associate a fixed reference nonrotating tetrad (local Lorentz frame of the observer) ${}^4_{(in)}E^A_{(\alpha)}$ with this inertial observer so that ${}^4_{(in)}E^A_{(o)}$ is his 4-velocity and the 4-acceleration vanishes [the space axes ${}^4_{(in)}E^A_{(a)}$ are defined modulo a rigid rotation]. For $s = s_o$, in the point $\sigma^A_o = \sigma^A(s_o) = \{\tau_o = \tau(s_o); \vec{\sigma}_o = \vec{\sigma}(s_o)\}$, let the tetrad ${}^4E^A_{(\alpha)}(\sigma(s))$ coincide with ${}^4_{(in)}E^A_{(\alpha)}$: ${}^4E^A_{(\alpha)}(\sigma_o) = {}^4_{(in)}E^A_{(\alpha)}$. For $s > s_o$ the evolution of the tetrad ${}^4E^A_{(\alpha)}(\sigma(s))$ may be parametrized as a Lorentz transformation with respect to ${}^4_{(in)}E^A_{(\alpha)}$: ${}^4E^A_{(\alpha)}(\sigma(s)) = {}^4_{(in)}E^A_{(\beta)} \Lambda^{(\beta)}{}_{(\alpha)}(s)$. It is assumed that the measures made with the clocks and rods of the accelerated observer are identical with those done by a unaccelerated momentarily comoving inertial observer with his clocks and rods; in Minkowski spacetime this is called the "locality hypothesis" in Ref. [94] and it applies also in general relativity, because, due to the equivalence principle, an observer in a gravitational field is equivalent to an accelerated observer in Minkowski spacetime.

Let $a^A(s) = \frac{du^A(s)}{ds}$, $a^A(s) u_A(s) = 0$, be the 4-acceleration of the accelerated observer. Among the tetrads ${}^4E^A_{(\alpha)}(\sigma(s))$ with ${}^4E^A_{(o)}(\sigma(s)) = u^A(s)$, the "nonrotating" one ${}^4_{(FW)}E^A_{(\alpha)}(\sigma(s))$ is the solution of the equations defining the "Fermi-Walker transport" (gyroscope-type transport) of a vector along the worldline Γ of the observer [see Ref. [71]; in this case the infinitesimal Lorentz transformation $\Lambda^{(\beta)}{}_{(\alpha)}(s) = \delta^{(\beta)}_{(\alpha)} + \omega^{(\beta)}{}_{(\alpha)}(s)$ generates only the appropriate Lorentz transformation in the timelike 2-hyperplane spanned by $u^A(s)$ and $a^A(s)$; under Fermi-Walker transport ${}^4E^A_{(o)}$ remains equal to u^A and the triad ${}^4E^A_{(a)}$ is the correct relativistic generalization of Newtonian nonrotating frames]

$$\frac{\delta}{\delta s}\quad {}^4_{(FW)}E^A_{(\alpha)}(\sigma(s)) = u^B(s)\, {}^4\nabla_B\, {}^4_{(FW)}E^A_{(\alpha)}(\sigma(s)) = -\Omega^A_{(FW)B}(s)\, {}^4_{(FW)}E^B_{(\alpha)}(\sigma(s)),$$
$$\Omega^{AB}_{(FW)}(s) = a^A(s) u^B(s) - a^B(s) u^A(s), \quad [\Omega^{AB}_{(FW)} w_B = 0 \quad if\ w_B u^B = w_B a^B = 0], \qquad \text{(A11)}$$

where $\frac{\delta}{\delta s}$ is the "absolute derivative" of the vector field restricted to the timelike worldline

Γ (its vanishing defines "parallel transport along Γ"). One speaks of "Fermi transport" of a vector F^A along Γ, if the vector is orthogonal to the 4-velocity u^A and it suffers Fermi-Walker transport, which reduces to $\frac{\delta}{\delta s} F^A(\sigma(s)) = u^A(s) a_B(s) F^B(\sigma(s))$ with $\frac{\delta}{\delta s} F^A u_A = 0$. Therefore ${}^4_{(FW)}E^A_{(a)}$ is said a "Fermi triad" and ${}^4_{(FW)}E^A_{(\alpha)}$ a "Fermi frame": this is the most natural generalization of an inertial reference frame along the path of an accelerated observer. In general, a Fermi frame cannot be extended to the whole spacetime manifold due to limitations imposed by curvature (tidal effects). The coordinated effort of many observers over an extended period of time can lead to a unique picture of natural phenomena (e.g. in astronomy) if these observers occupy a finite region of spacetime over which an extended nonrotating system can be defined [94]; in practice, however, the Newtonian framework is used for the sake of simplicity and relativistic effects are treated as small perturbations in a post-Newtonian approximation scheme.

For any other tetrad field one has $\frac{d}{ds}\, {}^4E^A_{(\alpha)}(\sigma(s)) = -\Omega^A{}_B(s)\, {}^4E^B_{(\alpha)}(\sigma(s))$ with $\Omega^{AB}(s) = \Omega^{AB}_{(FW)} + \Omega^{AB}_{(SR)}(s)$ with the spatial rotation part $\Omega^{AB}_{(SR)}(s) = \epsilon^{ABCD} u_C(s) \omega_D(s)$, $\omega^A u_A = 0$, producing a rotation in the spacelike 2-hyperplane perpendicular to u^A and ω^A [$\Omega^{AB}_{(SR)} u_B = \Omega^{AB}_{(SR)} \omega_B = 0$]. If at $s = s_1$ one has $u^A(s_1) = (1; \vec{0})$, $\omega^A(s_1) = (0; \vec{\omega})$, then $\frac{d}{ds}[{}^4E^r_{(a)} - {}^4_{(FW)}E^r_{(a)}](\sigma(s))|_{s=s_1} = \epsilon^{rst} \omega^s\, {}^4E^t_{(a)}(\sigma(s_1))$.

Given the tetrad ${}^4E^A_{(\alpha)}(\sigma(s))$ along the worldline Γ, the associated Frenet-Serret equations are [96]

$$
\begin{aligned}
&\frac{\delta}{\delta s}\, {}^4E^A_{(o)}(\sigma(s)) = \kappa(s)\, {}^4E^A_{(1)}(\sigma(s)),\\
&\frac{\delta}{\delta s}\, {}^4E^A_{(1)}(\sigma(s)) = \kappa(s)\, {}^4E^A_{(o)}(\sigma(s)) + \tau_1(s)\, {}^4E^A_{(2)}(\sigma(s)),\\
&\frac{\delta}{\delta s}\, {}^4E^A_{(2)}(\sigma(s)) = -\tau_1(s)\, {}^4E^A_{(1)}(\sigma(s)) + \tau_2(s)\, {}^4E^A_{(3)}(\sigma(s)),\\
&\frac{\delta}{\delta s}\, {}^4E^A_{(3)}(\sigma(s)) = -\tau_2(s)\, {}^4E^A_{(2)}(\sigma(s)),
\end{aligned}
\tag{A12}
$$

where $\kappa(s)$, $\tau_1(s)$, $\tau_2(s)$ are the curvature and the first and second torsion of Γ respectively [${}^4E^A_{(1)}$, ${}^4E^A_{(2)}$, ${}^4E^A_{(3)}$ are said the normal and the first and second binormal respectively].

In Ref. [71] [chapter 6 and section 13.6] there is the construction of the "proper reference frame" of an accelerated observer, which uses "Fermi normal coordinates" τ_F, $\vec{\sigma}_F$ [they are special Riemann normal coordinates which are normal in all the points of the 4-geodesic Γ]. This proper reference frame is both accelerated and rotating relative to the local Lorentz frames along Γ (as it can be shown with accelerometer measurements and from the rotation of inertial-guidance gyroscopes due to Coriolis and inertial forces). This proper reference frame can be extended around the worldline Γ till distances $l << \frac{c^2}{g}, \quad \frac{c}{\Omega}$ (the acceleration lengths for linear acceleration and rotation respectively [94]) due to inertial and tidal effects [the hypothesis of locality requires that the intrinsic length and time scales of the phenomena under observation be negligibly small relative to the corresponding acceleration scales associated with the observer].

The parameter τ_F is the proper time as measured by the accelerated observer's clock; the coordinates on the slice Σ_{τ_F} with normal l^A_F are the proper lengths (used as affine parameters) along 3-geodesics emanating from Γ (they are orthogonal to l^A_F and determine locally Σ_{τ_F}). The line element is [71,95]

$ds^2 = \epsilon \Big[[1 + {}^4R_{\tau r\tau s}\sigma^r_F\sigma^s_F](d\tau_F)^2 + \frac{4}{3}\, {}^4R_{\tau mrn}\sigma^m_F\sigma^n_F\, d\tau_F d\sigma^r_F - (\delta_{rs} - \frac{1}{3}\, {}^4R_{rsmn}\sigma^m_F\sigma^n_F) d\sigma^r_F d\sigma^n_F \Big] + O(|\vec{\sigma}_F|^3)$.

The observer carries with himself an orthonormal tetrad ${}^4E^A_{(\alpha)}$ with ${}^4E^A_{(o)} = l^A_F|_\Gamma = u^A$ (the 4-velocity of the observer), which changes from point to point of Γ: $\frac{\delta}{\delta s}\, {}^4E^A_{(\alpha)}(\sigma(s)) = -\Omega^A{}_B(s)\, {}^4E^B_{(\alpha)}(\sigma(s))$ [for $\omega^A = 0$ the observer would Fermi-Walker transport his tetrad (it would become a Fermi frame), while for $a^A = \omega^A = 0$ he would be freely falling (geodesic motion with local Lorentz frames along all Γ) with parallel transport of his tetrad: $u^B\, {}^4\nabla_B\, {}^4E^A_{(\alpha)} = 0$].

An accelerated observer looking at a freely falling particle as it passes through the origin of his proper reference frame [$v^r = \frac{dx^{(a)}}{dx^{(o)}}\, {}^3e^r_{(a)}$ is the 3-velocity of the particle; at the origin one chooses ${}^3E^A_{(o)} = u^A = (1; \vec{0})$, ${}^4E^A_{(a)} = (0; {}^3e^r_{(a)})$], sees the following 3-acceleration of the particle:

$\frac{d^2x^{(a)}}{dx^{(o)\,2}}\, {}^3e^r_{(a)} = -a^r - 2(\vec{\omega}\times\vec{v})^r + 2(\vec{a}\cdot\vec{v})v^r$,

where $a^A(0;\vec{a})$ is the observer's own 4-acceleration, $\vec{\omega}$ is the angular velocity with which his spatial triad ${}^3e^r_{(a)}$ is rotating. The three terms are the inertial acceleration, the Coriolis acceleration and a relativistic correction to the inertial acceleration respectively.

In particular the 3+1 splitting (slicing) with the spacelike hypersurfaces Σ_τ has the associated Σ_τ-adapted tetrads ${}^4_{(\Sigma)}\check{E}^A_{(\alpha)}$ of Eq.(40) with ${}^4_{(\Sigma)}\check{E}^A_{(o)} = l^A = \frac{\epsilon}{N}(1; -N^r)$: the unit normal vector field l^A to Σ_τ can be interpreted as the 4-velocity field of observers instantaneously at rest in the slices Σ_τ, called "Eulerian observers", because their motion follows the slices with 4-acceleration ${}^3a^A$ tangent to Σ_τ. For this special surface forming ($\omega_{AB} = 0$) nongeodesic congruence we have [we use the $\hat{b}^A_{\check{A}}$]

$$\begin{aligned}
{}^4\nabla_A l_B &= \epsilon\, {}^3a_B l_A - {}^3K_{AB},\\
\dot{l}_A &= {}^3a_A = {}^3a_r \hat{b}^r_A, \qquad {}^3a_r = \partial_r ln\, N,\\
\Theta &= {}^4\nabla_A l^A = \epsilon\, {}^3K,\\
\sigma_{AB} &= -[{}^3K_{rs} - \frac{1}{3} {}^3g_{rs}\, {}^3K]\hat{b}^r_A \hat{b}^s_B. \qquad \text{(A13)}
\end{aligned}$$

Let us remark that by a suitable choice of gauge it is possible to consider a local foliation whose leaves $\Sigma_{\tilde{\tau}}$ are orthogonal to a surface-forming timelike (or even spacelike) geodesic congruence [in general, this is possible only for a finite interval $\triangle\tau$, because coordinate singularities appear for increasing τ due to the focusing property of 4-geodesics]. This case corresponds to a local system of "Gaussian normal coordinates" [71] $\tilde{\tau}$, $\vec{\tilde{\sigma}}$ such that:

i) the shift functions vanish: $N^r = 0$ and ${}^4\tilde{g}_{\tilde{\tau}r} = 0$ [the surfaces $\Sigma_{\tilde{\tau}}$ are (locally) surfaces of simultaneity for the observers moving along the geodesics of the congruence];
ii) the coordinate time $\tilde{\tau}$ measures proper time along the geodesics: $d\tilde{\tau} = N d\tau$, $ds^2 = \epsilon \Big[(d\tilde{\tau})^2 - {}^3\tilde{g}_{rs} d\tilde{\sigma}^r d\tilde{\sigma}^s \Big]$.

These coordinates are also called "synchronous" coordinates; in cosmology, they are also said “comoving”, because the cosmological fluid (whose fluid lines are the geodesics of the congruence) is always at rest relative to $\Sigma_{\tilde{\tau}}$.

APPENDIX B: HAMILTONIAN EXPRESSION OF 4-TENSORS.

By using Eq.(57) and (65) to eliminate the τ-derivatives, we get the Hamiltonian version of the quantities defined in Appendix B [the symbol "$\circeq$" identifies the components of the 4-tensors whose phase space expression requires the first half of the Hamilton equations (65) for N, $N_{(a)}$, ${}^3e_{(a)r}$; remember that $\lambda_N \circeq \partial_\tau N$, $\lambda^{\vec N}_{(a)} \circeq \partial_\tau N_{(a)}$]. In this form we make explicit the dependence of 4-tensors on the arbitrary lapse and shift functions conjugate to the four first class constraints $\tilde \pi^N(\tau,\vec \sigma)\approx 0$, $\tilde \pi^{\vec N}_{(a)}(\tau,\vec \sigma)\approx 0$, but not yet the dependence on the further ten arbitrary functions conjugate to the remaining ten first class constraints, which have not yet been used in the expression of the 4-tensors. Let us remark that the 4-tensors of metric gravity do not depend on the three boost parameters $\varphi_{(a)}(\tau,\vec \sigma)$ [conjugate to $\tilde \pi^{\vec \varphi}_{(a)}(\tau,\vec \sigma)\approx 0$] and on the three angles conjugated to ${}^3\tilde M_{(a)}(\tau,\vec \sigma)\approx 0$.

We have [see Eqs.(38) for the expressions of ${}^3\Gamma^u_{rs}$, ${}^3\omega_{r(a)}$, ${}^3\Omega_{rs(a)}$, in terms of cotriads]

$$
\begin{aligned}
{}^4\Gamma^\tau_{\tau\tau} &= \frac{1}{N}\Big[\lambda_N + N_{(a)}\,{}^3e^r_{(a)}\partial_r N - \frac{\epsilon}{4k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}N_{(a)}N_{(b)}\,{}^3e_{(c)u}\,{}^3\tilde \pi^u_{(d)}\Big],\\
{}^4\Gamma^\tau_{r\tau} &= {}^4\Gamma^\tau_{\tau r} = \frac{1}{N}\Big[\partial_r N - \frac{\epsilon}{4k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}N_{(b)}\,{}^3e_{(c)u}\,{}^3\tilde \pi^u_{(d)}\Big],\\
{}^4\Gamma^\tau_{rs} &= {}^4\Gamma^\tau_{sr} = -\frac{1}{N}\frac{\epsilon}{4k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}\,{}^3e_{(b)s}\,{}^3e_{(c)u}\,{}^3\tilde \pi^u_{(d)},\\
{}^4\Gamma^u_{\tau\tau} &\circeq {}^3e^u_{(a)}\Big[\lambda^{\vec N}_{(a)} - \frac{N_{(a)}}{N}\lambda_N\Big] +\\
&\quad + N(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2}){}^3e^u_{(a)}\,{}^3e^v_{(b)}\partial_v N + {}^3e^v_{(b)}N_{(b)}({}^3e^u_{(a)}\,N_{(a)})_{|v} -\\
&\quad - \frac{\epsilon N}{2k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}N_{(a)}(\delta_{(b)(e)} - \frac{N_{(b)}N_{(e)}}{2N^2}){}^3e^u_{(e)}\,{}^3e_{(c)v}\,{}^3\tilde \pi^v_{(d)} -\\
&\quad - N_{(a)}\,{}^3e^u_{(b)}\Big[\frac{\epsilon N}{4k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(c)r}\,{}^3\tilde \pi^r_{(d)} +\\
&\quad + {}^3e^v_{(a)}(N_{(c)}\,{}^3e^r_{(c)}\partial_r\,{}^3e_{(b)v} + {}^3e_{(b)r}\,\partial_v(N_{(c)}\,{}^3e^r_{(c)})) +\\
&\quad + \epsilon_{(a)(b)(c)}\hat \mu_{(c)}\Big],\\
{}^4\Gamma^u_{r\tau} &= {}^4\Gamma^u_{\tau r} = {}^3e^u_{(a)}\Big[N_{(a)|r} - \frac{N_{(a)}}{N}\partial_r N\Big] -\\
&\quad - \frac{\epsilon N}{4k\,{}^3e}(\delta_{(a)(b)} - \frac{N_{(a)}N_{(b)}}{N^2}){}^3e^u_{(a)}\,{}^3G_{o(b)(c)(d)(e)}\,{}^3e_{(c)r}\,{}^3e_{(d)s}\,{}^3\tilde \pi^s_{(e)},\\
{}^4\Gamma^u_{rs} &= {}^4\Gamma^u_{sr} = {}^3\Gamma^u_{rs} + \frac{N_{(e)}}{N}\,\frac{\epsilon}{4k\,{}^3e}\,{}^3e^u_{(e)}\,{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(a)r}\,{}^3e_{(b)s}\,{}^3e_{(c)v}\,{}^3\tilde \pi^v_{(d)},
\end{aligned}
\tag{B1}
$$

$$
\begin{aligned}
{}^4\overset{\circ}{\omega}_{\tau(o)(a)} &= -{}^4\overset{\circ}{\omega}_{\tau(a)(o)} =\\
&= -\epsilon\,{}^3e^r_{(a)}\partial_r N - \frac{1}{4k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,N_{(b)}\,{}^3e_{(c)u}\,{}^3\tilde \pi^u_{(d)},\\
{}^4\overset{\circ}{\omega}_{\tau(a)(b)} &= -{}^4\overset{\circ}{\omega}_{\tau(b)(a)} \circeq -\epsilon\,[{}^3\omega_{r(a)(b)}\,{}^3e^r_{(c)}N_{(c)} + \epsilon_{(a)(b)(c)}\hat \mu_{(c)}],\\
{}^4\overset{\circ}{\omega}_{r(o)(a)} &= -{}^4\overset{\circ}{\omega}_{r(a)(o)} = -\frac{1}{4k\,{}^3e}{}^3G_{o(a)(b)(c)(d)}\,{}^3e_{(b)r}\,{}^3e_{(c)u}\,{}^3\tilde \pi^u_{(d)},\\
{}^4\overset{\circ}{\omega}_{r(a)(b)} &= -{}^4\overset{\circ}{\omega}_{r(b)(a)} = -\epsilon\,{}^3\omega_{r(a)(b)} = \frac{1}{2}\Big[{}^3e^s_{(a)}(\partial_r\,{}^3e_{(b)s} - \partial_s\,{}^3e_{(b)r}) +
\end{aligned}
$$

$$+ {}^3e^s_{(b)}(\partial_s {}^3e_{(a)r} - \partial_r {}^3e_{(a)s}) + {}^3e^u_{(a)} {}^3e^v_{(b)} {}^3e_{(c)r}(\partial_v {}^3e_{(c)u} - \partial_u {}^3e_{(c)v})\Big]. \qquad \text{(B2)}$$

$$\begin{aligned}
{}^4\overset{\circ}{\Omega}_{rs(a)(b)} &= {}^3e^u_{(a)} {}^3e^v_{(b)} {}^4R_{uvrs} = -\epsilon\Big[{}^3\Omega_{rs(a)(b)} + \\
&+ \frac{1}{(4k\,{}^3e)^2} {}^3G_{o(a)(c)(d)(e)} {}^3G_{o(b)(f)(g)(h)} \\
&\cdot\ ({}^3e_{(c)r} {}^3e_{(f)s} - {}^3e_{(c)s} {}^3e_{(f)r}){}^3e_{(d)u} {}^3\tilde{\pi}^u_{(e)} {}^3e_{(g)v} {}^3\tilde{\pi}^v_{(h)}\Big], \\
{}^4\overset{\circ}{\Omega}_{rs(o)(a)} &= \frac{1}{N} {}^3e^v_{(a)}({}^4R_{\tau vrs} - N^u\,{}^4R_{uvrs}) = \\
&= \frac{1}{4k} {}^3e^u_{(a)}\Big[(\frac{1}{{}^3e} {}^3G_{o(b)(c)(d)(e)} {}^3e_{(b)r} {}^3e_{(c)u} {}^3e_{(d)v} {}^3\tilde{\pi}^v_{(e)})_{|s} - \\
&- (\frac{1}{{}^3e} {}^3G_{o(b)(c)(d)(e)} {}^3e_{(b)s} {}^3e_{(c)u} {}^3e_{(d)v} {}^3\tilde{\pi}^v_{(e)})_{|r}\Big], \\
{}^4\overset{\circ}{\Omega}_{\tau r(a)(b)} &= {}^3e^u_{(a)} {}^3e^v_{(b)} {}^4R_{uv\tau r} \,\overset{\circ}{=} \\
&\overset{\circ}{=}\ -\epsilon\Big(\partial_\tau {}^3\omega_{r(a)(b)} + \frac{1}{2}(\epsilon_{(a)(b)(c)}\epsilon_{(d)(e)(f)} - \epsilon_{(a)(b)(d)}\epsilon_{(c)(e)(f)}) \cdot \\
&\quad {}^3e^s_{(c)}\Big[\frac{\epsilon N}{4k\,{}^3e} {}^3G_{o(d)(l)(m)(n)} {}^3e_{(l)s} {}^3e_{(m)v} {}^3\tilde{\pi}^v_{(n)} + \\
&+ N_{(l)} {}^3e^u_{(l)}\partial_u {}^3e_{(d)s} + {}^3e_{(d)u}\partial_s(N_{(l)} {}^3e^u_{(l)}) + \epsilon_{(d)(m)(n)}\hat{\mu}_{(m)} {}^3e_{(n)s} - \\
&- N_{(g)} {}^3e^u_{(g)}\partial_u {}^3e_{(d)s} - {}^3e_{(d)u}\partial_s(N_{(g)} {}^3e^u_{(g)})\Big]{}^3\omega_{r(e)(f)} + \\
&+ N_{(c)} {}^3e^s_{(c)} [{}^3\omega_s, {}^3\omega_r]_{(a)(b)} + \\
&+ \frac{\epsilon}{4k\,{}^3e} {}^3G_{o(c)(d)(e)(f)} {}^3e_{(c)r} {}^3e_{(e)u} {}^3\tilde{\pi}^u_{(f)}(\delta_{(a)(d)} {}^3e^u_{(b)} - \delta_{(b)(d)} {}^3e^u_{(a)})\partial_u N + \\
&+ \frac{1}{(4k\,{}^3e)^2}(\delta_{(a)(l)}\delta_{(b)(d)} - \delta_{(a)(d)}\delta_{(b)(l)}){}^3G_{o(d)(e)(f)(g)} {}^3G_{o(h)(l)(m)(n)} \cdot \\
&\cdot\ {}^3e_{(h)r}N_{(e)} {}^3e_{(f)w} {}^3\tilde{\pi}^w_{(g)} {}^3e_{(m)v} {}^3\tilde{\pi}^v_{(n)}\Big), \\
{}^4\overset{\circ}{\Omega}_{\tau r(o)(a)} &= \frac{1}{N} {}^3e^u_{(a)}({}^4R_{\tau u\tau r} - N^s\,{}^4R_{su\tau r}) \,\overset{\circ}{=} \\
&\overset{\circ}{=}\ -\epsilon\,{}^3e^s_{(a)}\Big[\partial_\tau {}^3K_{rs} + N_{|s|r} - \\
&- \frac{\epsilon}{4k\,{}^3e} {}^3G_{o(c)(d)(e)(f)} {}^3e_{(d)u} {}^3e_{(e)w} {}^3\tilde{\pi}^w_{(f)}\Big({}^3e_{(c)r}(N_{(b)} {}^3e^u_{(b)})_{|s} + {}^3e_{(c)s}(N_{(b)} {}^3e^u_{(b)})_{|r}\Big) - \\
&- \frac{\epsilon}{4k}N_{(b)} {}^3e^u_{(b)}\Big({}^3G_{o(c)(d)(e)(f)} {}^3e_{(c)s} {}^3e_{(d)u} {}^3e_{(e)w} {}^3\tilde{\pi}^w_{(f)}\Big)_{|r}\Big]. \qquad \text{(B3)}
\end{aligned}$$

In the last two equations the quantities $\partial_\tau {}^3\omega_{r(a)(b)}$ and $\partial_\tau {}^3K_{rs}$ are a shorthand for their expression which is given in Eqs.(66). Let us remark that, since $\partial_\tau {}^3K_{rs}$ depends on $\partial_\tau {}^3\tilde{\pi}^r_{(a)}$, the quantities ${}^4\overset{\circ}{\Omega}_{\tau r(o)(a)}$ [and, therefore, ${}^4R_{\tau r\tau s}$] are dynamical, because they require the use of the second half of the Hamilton equations (65) [i.e. of the Einstein equations ${}^4\bar{G}_{rs} \overset{\circ}{=} 0$] for their explicit phase space determination. Then, we get

$$\begin{aligned}
{}^4R_{rsuv} &= {}^3e_{(a)r} {}^3e_{(b)s} {}^4\overset{\circ}{\Omega}_{uv(a)(b)} = -\epsilon\Big[{}^3R_{rsuv} + \frac{1}{(4k\,{}^3e)^2} {}^3G_{o(a)(b)(c)(d)} {}^3G_{o(e)(f)(g)(h)} \\
&\cdot\ {}^3e_{(a)r} {}^3e_{(e)s}({}^3e_{(b)u} {}^3e_{(f)v} - {}^3e_{(b)v} {}^3e_{(f)u}) {}^3e_{(c)t} {}^3\tilde{\pi}^t_{(d)} {}^3e_{(g)w} {}^3\tilde{\pi}^w_{(h)}\Big],
\end{aligned}$$

$$
\begin{aligned}
{}^4R_{\tau ruv} &= {}^3e_{(a)u}\, {}^3e_{(b)v}\, {}^4\mathring{\Omega}_{\tau r(a)(b)} = N\, {}^3e_{(a)r}\, {}^4\mathring{\Omega}_{uv(o)(a)} + N_{(a)}\, {}^3e_{(b)r}\, {}^4\mathring{\Omega}_{uv(a)(b)}, \\
{}^4R_{\tau r\tau s} &= N\, {}^3e_{(a)r}\, {}^4\mathring{\Omega}_{\tau s(o)(a)} + N_{(a)}\, {}^3e_{(b)r}\, {}^4\mathring{\Omega}_{\tau s(a)(b)}.
\end{aligned}
\qquad \text{(B4)}
$$

By using Eqs.(A7) and (A8), we can get the phase space expression of ${}^4R_{AB} \,\mathring{=}\, 0$, ${}^4R \,\mathring{=}\,$, ${}^4C_{ABCD} \,\mathring{=}\, {}^4R_{ABCD}$. Let us remember that the acceleration of the integral curves with tangent vector $l^\mu(\tau, \vec{\sigma})$ [the normal to Σ_τ in $z^\mu(\tau, \vec{\sigma})$] is ${}^3a_r(\tau, \vec{\sigma}) = \partial_r\, ln\, N(\tau, \vec{\sigma})$.

REFERENCES

[1] P.A.M.Dirac, Can.J.Math. **2**, 129 (1950); "Lectures on Quantum Mechanics", Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, N.Y., 1964).

[2] L.Lusanna, "Solving Gauss' Laws and Searching Dirac Observables for the Four Interactions", talk at the "Second Conf. on Constrained Dynamics and Quantum Gravity", S.Margherita Ligure 1996, eds. V.De Alfaro, J.E.Nelson, G.Bandelloni, A.Blasi, M.Cavaglià and A.T.Filippov, Nucl.Phys. (Proc.Suppl.) **B57**, 13 (1997) (hep-th/9702114). "Unified Description and Canonical Reduction to Dirac's Observables of the Four Interactions", talk at the Int.Workshop "New non Perturbative Methods and Quantization on the Light Cone', Les Houches School 1997, eds. P.Grangé, H.C.Pauli, A.Neveu, S.Pinsky and A.Werner (Springer, Berlin, 1998) (hep-th/9705154). "The Pseudoclassical Relativistic Quark Model in the Rest-Frame Wigner-Covariant Gauge", talk at the Euroconference QCD97, ed. S.Narison, Montpellier 1997, Nucl.Phys. (Proc. Suppl.) **B64**, 306 (1998).

[3] S.Shanmugadhasan, J.Math.Phys. **14**, 677 (1973). L.Lusanna, Int.J.Mod.Phys. **A8**, 4193 (1993). M.Chaichian, D.Louis Martinez and L.Lusanna, Ann.Phys.(N.Y.)**232**, 40 (1994). L.Lusanna, Phys.Rep. **185**, 1 (1990); Riv. Nuovo Cimento **14**, n.3, 1 (1991); J.Math.Phys. **31**, 2126 (1990); J.Math.Phys. **31**, 428 (1990).

[4] P.A.M.Dirac, Can.J.Phys. **33**, 650 (1955).

[5] L.Lusanna, Int.J.Mod.Phys. **A10**, 3531 and 3675 (1995).

[6] L.Lusanna and P.Valtancoli, Int.J.Mod.Phys. **A12**, 4769 (1997), (hep-th/9606078).

[7] L.Lusanna and P.Valtancoli, Int.J.Mod.Phys. **A12**, 4797 (1997), (hep-th/9606079).

[8] L.Lusanna and P.Valtancoli, Int.J.Mod.Phys.**A13**, 4605 (1998) (hep-th/9707072).

[9] K.Kuchar, J.Math.Phys. **17**, 777, 792, 801 (1976); **18**, 1589 (1977).

[10] P.A.M.Dirac, *Rev.Mod.Phys.* **21** (1949) 392.

[11] L.Lusanna, Int.J.Mod.Phys. **A12**, 645 (1997).

[12] M.Pauri and M.Prosperi, J.Math.Phys. **16**, 1503 (1975).

[13] W.G.Dixon, J.Math.Phys. **8**, 1591 (1967). "Extended Objects in General Relativity: their Description and Motion", in "Isolated Gravitating Systems in General Relativity", ed. J.Ehlers (North-Holland, Amsterdam, 1979).

[14] G.Longhi and M.Materassi, "A Canonical Realization of the BMS Algebra", J.Math.Phys. **40**, 480 (1999) (hep-th/9803128); "Collective and Relative Variables for a Classical Klein-Gordon Field", Firenze Univ. preprint 1998 (hep-th/9890024).

[15] D.Alba, L.Lusanna and M.Pauri, in preparation.

[16] L.Lusanna and M.Materassi, "The Canonical Decomposition in Collective and Relative Variables of a Klein-Gordon Field in the Rest-Frame Wigner-Covariant Instant Form", Firenze Univ. preprint (hep-th/9904202).

[17] D.Alba and L.Lusanna, Int.J.Mod.Phys.**A13**, 2791 (1998) (hep-th/9705155).

[18] D.Alba and L.Lusanna, Int.J.Mod.Phys.**A13**, 3275 (1998) (hep-th/9705156).

[19] F.Bigazzi and L.Lusanna, Int.J.Mod.Phys. **A14**, 1429 (1999), (hep-th/9807052).

[20] F.Bigazzi and L.Lusanna, Int.J.Mod.Phys. **A14**, 1877 (1999), (hep-th/9807054).

[21] C.Lämmerzahl, J.Math.Phys. **34**, 3918 (1993).

[22] I.Herbst, Commun.Math.Phys. **53**, 285 (1977); **55**, 316 (1997).
B. and L. Durand, Phys.Rev. **D28**, 396 (1983); erratum Phys.Rev. **D50**, 6642 (1994).

J.J.Basdevant and S.Boukraa, Z.Phys. **C28**, 413 (1985).
A.Martin and S.M.Roy, Phys.Lett. **B233**, 407 (1989).
A.LeYaouanc, L.Oliver and J.C.Raynal, Ann.Phys.(N.Y.) **239**, 243 (1995).
W.Lucha and F.F.Schöberl, Phys.Rev. **D50**, 5443 (1994).

[23] G.Leibbrandt, "Non-Covariant Gauges", ch.9 (World Scientific, Singapore, 1994).

[24] C.Møller, Ann.Inst.H.Poincaré **11**, 251 (1949); "The Theory of Relativity" (Oxford Univ.Press, Oxford, 1957).

[25] G.Veneziano, "Quantum Strings and the Constants of Nature", in "The Challenging Questions", ed.A.Zichichi, the Subnuclear Series n.27 (Plenum Press, New York, 1990).

[26] P.G.Bergmann, Rev.Mod.Phys. **33**, 510 (1961).

[27] C.J.Isham, "Canonical Quantum Gravity and the Problem of Time", in "Integrable Systems, Quantum Groups and Quantum Field Theories", eds.L.A.Ibort and M.A.Rodriguez, Salamanca 1993 (Kluwer, London, 1993); "Conceptual and Geometrical Problems in Quantum Gravity", in "Recent Aspects of Quantum Fields", Schladming 1991, eds. H.Mitter and H.Gausterer (Springer, Berlin, 1991); "Prima Facie Questions in Quantum Gravity" and "Canonical Quantum Gravity and the Question of Time", in "Canonical Gravity: From Classical to Quantum", eds. J.Ehlers and H.Friedrich (Springer, Berlin, 1994).

[28] K.Kuchar, "Time and Interpretations of Quantum Gravity", in Proc.4th Canadian Conf. on "General Relativity and Relativistic Astrophysics", eds. G.Kunstatter, D.Vincent and J.Williams (World Scientific, Singapore, 1992).

[29] M.Nakahara, "Geometry, Topology and Physics" (IOP, Bristol, 1990).

[30] B.O'Neil, "Semi-Riemannian Geometry" (Academic Press, New York, 1983).

[31] D.Bleecker, "Gauge Theory and Variational Principles" (Addison-Wesley, London, 1981).

[32] R.Arnowitt, S.Deser and C.W.Misner, Phys.Rev. **117**, 1595 (1960); in "Gravitation: an Introduction to Current Research", ed.L.Witten (Wiley, New York, 1962).

[33] T.Regge and C.Teitelboim, Ann.Phys.(N.Y.) **88**, 286 (1974).

[34] R.Beig and Ó Murchadha, Ann.Phys.(N.Y.) **174**, 463 (1987).

[35] L.Andersson, J.Geom.Phys. **4**, 289 (1987).

[36] T.Thiemann, Class.Quantum Grav. **12**, 181 (1995).

[37] A.Ashtekar, "Asymptotic Structure of the Gravitational Field at Spatial Infinity", in "General Relativity and Gravitation", Vol. 2, ed.A.Held (Plenum, New York, 1980). A.Ashtekar and R.O.Hansen, J.Math.Phys. **19**, 1542 (1978). A.Ashtekar and A.Magnon, J.Math.Phys. **25**, 2682 (1984). A.Ashtekar and J.D.Romano, Class.Quantum Grav. **9**, 1069 (1992).

[38] R.M.Wald, "General Relativity" (Chicago Univ.Press, Chicago, 1984).

[39] R.Geroch, J.Math.Phys. **9**, 1739 (1968); **11**, 343 (1970).

[40] A.Sen, Int.J.Theor.Phys. **21**, 1 (1982).
P.Sommers, J.Math.Phys. **21**, 2567 (1980).

[41] A.Ashtekar, "New Perspectives in Canonical Gravity" (Bibliopolis, Napoli, 1988).

[42] Y.Choquet-Bruhat and J.W.York jr., "The Cauchy Problem", in "General Relativity and Gravitation", vol.1, ed. A.Held (Plenum, New York, 1980).

[43] J.A.Wheeler, "Geometrodynamics and the Issue of the Final State", in "Relativity, Groups and Topology", Les Houches 1963, eds. B.S.De Witt and C.De Witt (Gordon and

Breach, London, 1964). "Superspace and the Nature of Quantum Geometrodynamics", in Battelle Rencontres 1967, eds. C.De Witt and J.A.Wheeler (Gordon and Breach, New York, 1968).

[44] A.E.Fischer, "The Theory of Superspace", in "Relativity", eds. M.Carmeli, L.Fickler and L.Witten (Plenum, New York, 1970); Gen.Rel.Grav. **15**, 1191 (1983); J.Math.Phys. **27**, 718 (1986).
M.Rainer, "The Moduli Space of Local Homogeneous 3-Geometries", talk at the Pacific Conf. on Gravitation and Cosmology, Seoul 1996.

[45] S.Timothy Swift, J.Math.Phys. **33**, 3723 (1992); **34**, 3825 and 3841 (1993).

[46] J.M.Arms, J.E.Marsden and V.Moncrief, Commun.Math.Phys. **78**, 455 (1981).

[47] D.Giulini, Helv.Phys.Acta **68**, 86 (1995).

[48] J.Lee and R.M.Wald, J.Math.Phys. **31**, 725 (1990).

[49] R.Beig, "The Classical Theory of Canonical General Relativity", in "Canonical Gravity: From Classical to Quantum", Bad Honnef 1993, eds. J.Ehlers and H.Friedrich, Lecture Notes Phys. 434 (Springer, Berlin, 1994).

[50] A.Lichnerowicz, J.Math.Pure Appl. **23**, 37 (1944).
Y.Choquet-Bruhat, C.R.Acad.Sci.Paris **226**, 1071 (1948); J.Rat.Mech.Anal. **5**, 951 (1956); "The Cauchy Problem" in "Gravitation: An Introduction to Current Research", ed.L.Witten (Wiley, New York, 1962).

[51] J.W.York jr, Phys.Rev.Lett. **26**, 1656 (1971); **28**, 1082 (1972). J.Math.Phys. **13**, 125 (1972); **14**, 456 (1972). Ann.Ins.H.Poincaré **XXI**, 318 (1974).
N.O'Murchadha and J.W.York jr, J.Math.Phys. **14**, 1551 (1972). Phys.Rev. **D10**, 428 (1974).

[52] J.W.York jr., "Kinematics and Dynamics of General Relativity", in "Sources of Gravitational Radiation", Battelle-Seattle Workshop 1978, ed.L.L.Smarr (Cambridge Univ.Press, Cambridge, 1979).

[53] I.Ciufolini and J.A.Wheeler, "Gravitation and Inertia" (Princeton Univ.Press, Princeton, 1995).

[54] A.Qadir and J.A.Wheeler, "York's Cosmic Time Versus Proper Time", in "From SU(3) to Gravity", Y.Ne'eman's festschrift, eds. E.Gotsma and G.Tauber (Cambridge Univ.Press, Cambridge, 1985).

[55] P.Havas, Gen.Rel.Grav. **19**, 435 (1987).
R.Anderson, I.Vetharaniam and G.E.Stedman, Phys.Rep. **295**, 93 (1998).

[56] L.Lusanna and S.Russo, "Tetrad Gravity: II) Dirac's Observables", Firenze Univ. preprint (gr-qc/9807073).

[57] J.Butterfield and C.J.Isham, "Spacetime and the Philosophical Challenge of Quantum Gravity", Imperial College preprint (gr-qc/9903072).

[58] H.Weyl, Z.Physik **56**, 330 (1929).

[59] P.A.M.Dirac, in "Recent Developments in General Relativity", (Pergamon Press, Oxford, and PWN-Polish Scientific Publishers, Warsaw, 1962).

[60] J.Schwinger, Phys.Rev. **130**, 1253 (1963).

[61] T.W.B.Kibble, J.Math.Phys. **4**, 1433 (1963).

[62] S.Deser and C.J.Isham, Phys.Rev. **D14**, 2505 (1976).
J.E.Nelson and C.Teitelboim, Ann.Phys.(N.Y.) **116**, 86 (1978).
M.Pilati, Nucl.Phys. **B132**, 138 (1978).

L.Castellani, P.van Nieuwenhuizen and M.Pilati, Phys.Rev. **D26**, 352 (1982).
J.E.Nelson and T.Regge, Ann.Phys.(N.Y.) **166**, 234 (1986); Int.J.Mod.Phys. **A4**, 2021 (1989).

[63] J.M.Charap and J.E.Nelson, J.Phys. **A16**, 1661 and 3355 (1983). Class.Quantum Grav. **3**, 1061 (1986).
J.M.Charap, "The Constraints in Vierbein General Relativity", in "Constraint's Theory and Relativistic Dynamics", eds. G.Longhi and L.Lusanna (World Scientific, Singapore, 1987).

[64] M.A.Clayton, Class.Quantum Grav. **14**, 1851 (1997); J.Math.Phys. **39**, 3805 (1998).

[65] J.W.Maluf, Class.Quantum Grav. **8**, 287 (1991).

[66] M.Henneaux, Gen.Rel.Grav. **9**, 1031 (1978).
J.Geheniau and M.Henneaux, Gen.Rel.Grav. **8**, 611 (1977).

[67] M.Henneaux, Phys.Rev. **D27**, 986 (1983).

[68] J.M.Charap, M.Henneaux and J.E.Nelson, Class.Quantum Grav. **5**, 1405 (1988).

[69] M.Henneaux, J.E.Nelson and C.Schonblond, Phys.Rev. **D39**, 434 (1989).

[70] R.Schmidt, "Infinte Dimensional Hamiltonian Systems" (Bibliopolis, Napoli, 1984).

[71] C.W.Misner, K.S.Thorne and J.A.Wheeler, Gravitation (Freeman, New York, 1973).

[72] S.Weinberg, "Gravitation and Cosmology" (J.Wiley, New York, 1972).

[73] J.Isenberg and J.Nester, "Canonical Gravity", in "General Relativity and Gravitation", vol.1, ed.A.Held (Plenum, New York, 1980).

[74] B.Carter, J.Geom.Phys. **8**, 53 (1992).

[75] W.R.Davies, "Classical Fields, Particles and the Theory of Relativity" (Gordon and Breach, New York, 1970).

[76] J.W.York jr., Found.Phys. **16**, 249 (1986).

[77] J.B.Romano, Gen.Rel.Grav. **25**, 759 (1993).

[78] B.S.De Witt, Phys.Rev. **160**, 1113 (1967).

[79] A.E.Fischer and J.E.Marsden, "The Initial Value Problem and the Dynamical Formulation of General Relativity", in "General Relativity. An Einstein Centenary Survey", eds. S.W.Hawking and W.Israel (Cambridge Univ.Press, Cambridge, 1979). Y.Choquet-Bruhat, A.Fischer and J.E.Marsden, "Maximal Hypersurfaces and Positivity of Mass", LXVII E.Fermi Int.School of Physics "Isolated Gravitating Systems in General Relativity", ed. J.Ehlers (North-Holland, Amsterdam, 1980).

[80] B.S.De Witt, Phys.Rev. **162**, 1195 (1967); "The Dynamical Theory of Groups and Fields" (Gordon and Breach, New York, 1967) and in "Relativity, Groups and Topology", Les Houches 1963, eds. C.De Witt and B.S.De Witt (Gordon and Breach, London, 1964); "The Spacetime Approach to Quantum Field Theory", in "Relativity, Groups and Topology II", Les Houches 1983, eds. B.S.DeWitt and R.Stora (North-Holland, Amsterdam, 1984).

[81] S.W.Hawking and G.T.Horowitz, Class.Quantum Grav. **13**, 1487 (1996).

[82] K.Kuchar, "Canonical Quantum Gravity" in "General Relativity and Gravitation" Int.Conf. GR13, Cordoba (Argentina) 1992, eds. R.J.Gleiser, C.N.Kozameh and O.M.Moreschi (IOP, Bristol, 1993).

[83] G.Longhi and L.Lusanna, Phys.Rev. **D34**, 3707 (1986).

[84] A.López-Pinto, A.Tiemblo and R.Tresguerres, Class.Quantum Grav. **14**, 549 (1997), (gr-qc/9603023).

[85] J.N.Goldberg, Phys.Rev. **D37**,2116 (1988).
[86] C.Teitelboim, “The Hamiltonian Structure of Space-Time”, in “General Relativity and Gravitation”, ed.A.Held, Vol.I (Plenum, New York, 1980). A.S.Hojman, K.Kuchar and C.Teitelboim, Ann.Phys. (N.Y.) **96**, 88 (1971).
[87] J.A.Wheeler, in “Relativity, Groups and Topology”, Les Houches 1963, eds. C.De Witt and B.S.De Witt (Gordon and Breach, New York, 1964).
[88] F.Antonsen and F.Markopoulou, “4D Diffeomorphisms in Canonical Gravity and Abelian Deformations”, Imperial/TP/96-97/26 (gr-qc/9702046).
[89] R.Sugano, Y.Kagraoka and T.Kimura, Int.J.Mod.Phys. A7, 61 (1992).
[90] J.Stachel, in “General Relativity and Gravitation”, GR11, Stockholm 1986, ed. M.A.H.Mac Callum (Cambridge Univ. Press, Cambridge, 1987); “The Meaning of General Covariance”, in “Philosophical Problems of the Internal and External Worlds”, Essays in the Philosophy of A.Grünbaum, eds. J.Earman, A.I.Janis, G.J.Massey and N.Rescher (Pittsburgh Univ.Press, Pittsburgh, 1993).
C.Rovelli, Class.Quantum Grav. **8**, 297 and 317 (1991).
[91] H. van Elst and C.Uggla, Class.Quantum Grav. **14**, 2673 (1997).
[92] R.T.Jantzen, P.Carini and D.Bini, Ann.Phys.(N.Y.) **215**, 1 (1992).
[93] S.Boersma and T.Dray, Gen.Rel.Grav. **27**, 319 (1995).
[94] B.Mashhoon, Phys.Lett. **143A**, 176 (1990) and **145A**, 147 (1990); Phys.Rev. **47A**, 4498 (1993).
[95] F.K.Manasse and J.A.Wheeler, J.Math.Phys. **4**, 735 (1963).
[96] E.Honig, E.L.Schucking and C.V.Vishveshwara, J.Math.Phys. **15**, 774 (1974). B.R.Iyer and C.V.Vishveshwara, Class.Quantum Grav. **5**, 961 (1988) ; Phys.Rev. **D48**, 5706 (1993).